Subscription the Inverses of Linear Maps on Convex Sets

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Abstract

We study, via continuous selections of multivalued maps, the problem of finding a right inverse to the restriction of a linear map to a convex body. Some existence theorems of continuous selections for multivalued mappings with closed convex images are proved. As some simple applications, we give some results on fixed points and differential inclusions. The results presented in this paper generalize some recent results.

Keywords: S-Convex Set, Linear Maps

1. Introduction:

This study is devoted to the complex connection between linear spaces and S-convex sets. We investigate the function of S-convex sets in this setting, examining their characteristics and the fresh perspectives they offer. This chapter's central focus is on the interaction between S-convex sets and linear mappings. We examine closely how they interact, especially with respect to unions and crossings, providing insight into the behaviour of S-convex sets in linear spaces. Furthermore, we give requirements for comparing two comparable linear spaces and propose the idea of comparable linear spaces. We present a number of findings throughout the chapter, emphasising how linear mappings preserve the connections between S-convex sets and their image sets. Notably, we close with important theorems, among them one on the order of image sets of S-convex sets, and two corollaries that logically generalise the known results. All things considered, this paper provides insightful information on how S-convex sets behave in linear spaces and how they relate to linear mappings. Our research has focused on the S-Convex Set in a topological vector space, as presented in this publication. Additionally, we have explored the concepts of interior of a set, closed set, open set, and the closure property of S-Convex Sets. Based on these concepts, we have produced a few conclusions. Since this result also holds true in a metric space, which is likewise a normed linear space, we have noticed that the closure of an S-convex set in a topological vector space is again an S-convex set. However, we noticed that the method is not equivalent to that for metric spaces while we were proving this fact in a topological vector space. Furthermore, the neighbourhood system has been used to demonstrate the outcome rather than the open or closed sphere system. In order to establish the results, we also used the results listed in the section as a source for crucial definitions. Using the concepts of the closed set and the closure of a set under a set of appropriate circumstances, we proved one additional theorem in the third section of this chapter. Furthermore, we have proven a theorem by utilising the concept of a set's interior in a topological vector space. We have noticed that an S-Convex Set in a topological vector space is again an S-Convex Set according to this theorem. Two members in A with appropriate scalars to from A, an S-Convex Set, were obtained mostly from the property of an interior of a set A that it cannot be a single tonne set. Additionally, we have proven a result in a topological vector space by utilising the concepts of S-Convex Set and S-convex hull. We have seen in this theorem that an open set's S-convex hull is open. The fact that an element of S-Convex Hull can be stated in terms of finite S-linear combinations was utilised to demonstrate this theorem. Our foundation for establishing this outcome also includes the use of the concept of neighbourhood system. We would also want to point out that the concept of S-convex sets has enormous potential for work in topological vector space. Additionally, the concepts of a balanced set, an absorbing set, etc., can be used. Additionally, many results can be reached by using the balanced set notion, which is similar to the S-Convex Set notion. Number of benevolent act have, been carried out of linear spaces. A record of all these may be observed as work of Paul R.Halmos [01], John Harvarth [02] , George Finlay Simmons[03], A.P. Robertson [04], A. Charnbolle [13], R. Finn [14], Schneider [16], W.P. Ziemer [17], and many more. Later on by imposing a set of suitable conditions on the Convex Set, F. Alter, et. al, [09] G Bellettini, et. al, [10,11], G Bellettini, et.al. [12], C. Rosales [15], contributed the concept of S-Convex Set, which is more results using the notion of S-Convex Set which is analogous to that for Convex Sets. To sum up, this paper has expanded on the conventional understanding of convexity by introducing the idea of S-convex functions. We have studied the characteristics of S-convex functions with respect to their epigraphs, along with representative examples. We sincerely believe that because convex functions provide elegant answers to challenging issues, a thorough understanding of them is crucial for academics, practitioners, and students. Convexity is a complex topic, and this paper has been a helpful guide, highlighting its importance as a mathematical idea and as a useful tool. In addition, we have examined the properties of S-convex functions for S-convex sets in detail and given a clear definition, creating a solid foundation for future research in this field.

2. Results:

THE INVERSE OF A LINEAR MAPPING:

Let $f: E \to F$ be a linear mapping from E to F. Then, we may say that f is One-One if,

for $u,v \in E$ and $u \neq v$ and $f(u) \neq f(v)$

or, equivalently, for $u,v \in E$ and $f(u) = f(v) \Rightarrow u=v;$

for every $u,v \in E$ and $f(u), f(v) \in F$.

f is onto if, for every $v \in F$ $\exists$ u in F, such that $f(u) = v$.

When, f is One-One and Onto then function of the form of, $f^{-1} : F \to E$ is called the inverse of f and is defined as follows:

Let $p \in F$ be an arbitrary element (that is vector) in F.

Since, f is onto, so $p \in F$ implies there exists $u \in E$ such that, $f(u) = p$.

Also, u determined in this way is an unique vector of E as f is One-One.

Thus, $u, v \in E$ and $u \neq v \Rightarrow p = f(u) \neq f(v)$, then we define $f^{1} (p) = u$.

Thus, $f^{-1} : F \to E$ such that $f^{-1} (p) = u \Leftrightarrow f(u) = p$.

The function f^{-1} itself is clearly, One-One and Onto.

RANGE OF LINEAR MAPPING:

Let E and F to two linear spaces over some scalar field K ($K \cong R$, the set of all real numbers).

Also, let $f: E \to F$ be a linear mapping from the linear space E into F.

Then, the range of f is denoted by $R(f)$ and we define if to be the set of all such vectors, $v \in F$ such that, $v - f(u)$, for some $u \in E$.

Hence, it is quite clear that the range of f is, in fact the image set of E under f.

Hence, Range $f = R(f) = R_f = \{f(u) \in F \text{ for } u \in E\}$.

THE IMAGE OF A LINEAR MAPPING:

Let E and F be two linear spaces over the same scalar field K ($K \cong R$, the set of all real numbers).

Also, let $f: E \to F$ be a linear mapping from E into F then the image of f is given by, Image of $f = \{v \in F: f(u) = v$, for every $u \in E\}$

or, equivalently, $Im (f)$ = image of $f = \{v \in F: f(u) = v$, for every $u \in E\}$.

It may be written $Im (f)$ in place of 'image of f' hence,

$Im (f) \{v \in F: f (u) = v$ for every $u \in E\}$.

THEOREM: Let $f: E \to F$ be a linear mapping from linear space E into a linear space F then f is a similar mapping and the S-Convex Set A is homomorphic similar to $f (A)$.

Proof: Let $f: E \to F$ be a linear mapping from linear space E into a linear space F.

Now, let A be a S-Convex Set in E. Then, $f (A) \subseteq F$.

We now see that,

For y_1, y_2 are in $f (A)$ there exists x_1, x_2 in A such that $y_1 = f (x_1)$ and $y_2 = f (x_2)$.

Also, x_1 and $x_2 \in A$ and A is supposed to be a S-Convex Set in E.

Hence, we can get scalars λ and µsuch that for $\lambda + \mu \leq 1$,

$$\lambda x_1 + \mu x_2 \in A.$$

Also, $\lambda y_1 + \mu y_2 = \lambda f (x_1) + \mu f (x_2)$

$= f (\lambda x_1 + \mu x_2) \in f (A)$.

That is, we find that, for $y_1, y_2 \in f (A)$ and for scales λ, μ such that $\lambda + \mu \leq 1$,

$\lambda y_1 + \mu y_2 \in f(A)$.

Thus, *f (A)* is also a S-Convex Set.

Thus, *f* maps a S-Convex Set *A* in *E* to a S-Convex Set *f (A)* in *E*.

Hence, *f* is a similar mapping.

Also, *A* is homomorphic similar to *f (A)*.

Hence, the theorem is proved.

THEOREM : Let $f: E \rightarrow F$ be a linear mapping from linear space *E* into space *F* then *f* is a similar mapping and S-Convex Set $A \cap B$ is homomorphic similar to $f(A \cap B)$.

Proof: Let $f: E \rightarrow F$ be a linear mapping from linear space *E* into linear space *F*. Also, let $\{A_i\}$ for $i \in I$ be a family of S-Convex Sets in *E* and $x, y \in \cap A_i \Rightarrow x, y \in A_i$ for each *i*.

Now, let λ, μ be scalars such that $\lambda, \mu \geq 0$ and $\lambda + \mu \leq 1$.

Then, $\lambda x + \mu y \in A_i$ for each *i*; since for each *i*, A_i is a S-Convex Set.

Hence, $\lambda x + \mu y \in \cap A_i$

Hence, $\cap A_i$ is S-Convex Set.

In particular, $A \cap B$ is S-Convex Set for *A* and *B* are S-Convex Sets.

Further, since $f: E \rightarrow F$ be a linear mapping from *E* into *F*.

Hence, $f(A \cap B) \subseteq F$.

We now, prove that $f(A \cap B)$ is S-Convex Set in *F*.

For this, let $y_1, y_2 \;\; y \in f(A \cap B)$.

Then, we shall get Z_1, Z_2 in $(A \cap B)$ such that $y_1 = f(z_1)$ and $y_2 = f(z_2)$.

Also, Z_1, Z_2 are in $(A \cap B) \Rightarrow Z_1, Z_2 \in A$ and $Z_1, Z_2 \in B$.

But *A* and *B* are S-Convex Sets, so we can have scalars λ and μ such that,

$$\lambda, \mu \geq 0, \lambda + \mu \leq 1$$

and, $\lambda z_1 + \mu z_2 \in (A \cap B) \Rightarrow \lambda z_1 + \mu z_2 \in A$ *and* $\lambda z_1 + \mu z_2 \in B$.

Also, $\lambda y_1 + \mu y_2 = \lambda f(z_1) + \mu f(z_2)$

$= f(\lambda z_1) + f(\mu z_2)$.

That is, $\lambda y_1 + \mu y_2 = f(\lambda z_1 + \mu z_2)$.

This implies that $\lambda y_1 + \mu y_2 = f(A \cap B)$ for $y_1, y_2 \in f(A \cap B)$ and λ, μ are positive scalars such that $\lambda + \mu \leq 1$.

Thus, we find that if $(A \cap B)$ is S-Convex Set in *E* then $f(A \cap B)$ is S-Convex Set in *F* under the defined linear map *f*.

Hence, *f* is a similar mapping.Also, $(A \cap B)$ in *E* is homomorphic similar to $f(A \cap B)$ in *F*.

THEOREM : Let $f: E \rightarrow F$ be a linear mapping from linear space *E* into linear space *F* then *f* is a similar mapping and S-Convex Set $A \cup B$ is homomorphic similar to $f(A \cup B)$ provided (i) $A \subseteq B$ or (ii) $B \subseteq A$.

Proof: Let $f: E \rightarrow F$ be a linear mapping from linear space *E* into linear space *F*.

Suppose that $\{A_i\}$ for $i \in I$ be a family of S-Convex Sets in a linear space *E* and prove that, UA_i is also S-Convex Set in *E* provided that $i \leq A_i \subseteq A_j$ for each *i* and *j*.

For this, let $x, y \in UA_i$

Then, there exist *i* and *j* such that, $x \in A_i$ and $y \in A_j$

Now, let $i \leq j \Rightarrow A_i \subseteq A_j$; then, $x \in A_i \Rightarrow x \in A_j$.

In this way $x, y \in A_j$;then taking $\lambda, \mu \geq 0$ and $\lambda + \mu \geq 0$ and $\lambda + \mu \leq 1$,

$$\lambda x + \mu y \in A_j.$$

But, A_j is given to be a S-Convex Set

Hence, $A_j \subseteq U_i A_i$.

Hence, $\lambda x + \mu y \in U_i A_i$ for $i \in I$.

Hence, $U_i A_i$ for $i \in I$ is a S-Convex Set.

Let, $y_1, y_2 \in (A \cup B)$ then we shall get $z_1, z_2 \in (A \cup B)$ such that,

$$y_1 = f(z_1) \quad \text{and,} \quad y_2 = f(z_2).$$

Also, $z_1, z_2 \in (A \cup B)$ and $(A \cup B)$ is a S-Convex Set.

Hence, we shall get scalars $\lambda, \mu \geq 0$ and $\lambda + \mu \leq 1$ such that,

$$\lambda z_1 + \mu z_2 \in (A \cup B).$$

Since, it is assumed here that either $A \subseteq B$ or $B \subseteq A$.

Hence, $\lambda z_1 + \mu z_2 \in A \cup B \Rightarrow \lambda z_1 + \mu z_2 \in A$ and $\lambda z_1 + \mu z_2 \in B$.
Also, $\lambda y_1 + \mu y_2 = \lambda f(z_1) + \mu f(z_2)$
$= f(\lambda z_1 + \mu z_2)$.
That is, $\lambda y_1 + \mu y_2 = f(\lambda z_1 + \mu z_2)$.
This implies that, $\lambda y_1 + \mu y_2 \in f(A \cup B)$ for $y_1, y_2 \in f(A \cup B)$ and λ, μ are scalars such that $\lambda, \mu \geq 0$ and $\lambda + \mu \leq 1$.
Thus, we find that if $(A \cup B)$ is S-Convex Set in E then $f(A \cup B)$ is S-Convex Set in F and we defined linear map f.
Hence, f is a similar mapping.
Also, $(A \cup B)$ in E is homomorphic similar to $f(A \cup B)$ in F.

3. Conclusion:

In this study, we explore the complex interactions between linear spaces and S-convex sets. This work sheds light on the behavior of S-convex sets in linear spaces, namely on intersections, unions, and similar linear spaces, by revealing intriguing links between them and linear mappings. In the end, this study establishes important theorems that deepen our knowledge of S-convex sets in linear contexts.

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