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Fractional Derivatives and Generating Functions Involving Hyper Geometric Series of Three Variables

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Abstract

Generating functions plays an essential role in the investigation of several useful properties of the sequences which they generate. In this paper, we establish certain generating relations, involving some quadruple hyper geometric functions introduced by Bin-Saad and Younis. Some interesting special cases of our main results are also considered

Keywords: Differential Equation, hyper geometric

1. INTRODUCTION

The hypergeometric series is the most useful and important special function, and it has been studied to solve various problems in many areas of mathematics, physics, statistics, and engineering. Hypergeometric series in several variables appear in numerous fields of applied mathematics, mathematical physics, and chemistry. When it becomes an arbitrary parameter, the nth derivative and n-fold integral are of interest in the theory of FC. S.F. Lacroix gave the m th derivative to be starting from $y = xn$, where n is a positive integer.

$$\frac{d^m y}{dx^m} = \frac{n!}{(n-m)!} x^{n-m}$$

He arrived at the formula by using the GMF as normal, substituting Yi form, and any positive real integer for n.

$$\frac{d^{1/2} y}{dx^{1/2}} = \frac{\Gamma(a+1)}{\Gamma\left(a+\frac{1}{2}\right)} x^{a-\frac{1}{2}}$$

He gave the example for $y = x$ and derived.

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}(x) = \frac{\Gamma(2)}{\Gamma\left(\frac{3}{2}\right)} x^{\frac{1}{2}} = \frac{2\sqrt{x}}{\sqrt{\pi}}.$$

The current Rieman Liouvill definition of a FC also produces this conclusion.
The FD of order v is defined by Liouvill as

$$D_x^v f(x) = \sum_{n=0}^{\infty} c_n a_n^v e^{o_* x}$$

$$D_x^v f(x) = \sum_{n=0}^{\infty} c_n a_n^v e^{a_* x} (1)$$

where

$$f(x) = \sum_{n=0}^{\infty} c_n e^{v_n x} (2)$$

When dealing with explicit FCS of the type $x - a, a > 0$, Liovil1e's second technique was used. He thinks the essential

$$I = \int_0^{\infty} u^{a-1} e^{-xu} du$$

Then with the use of (1) he obtained the following outcome

$$D_x^v x^{-a} = \frac{(-1)^v \Gamma(a+v)}{\Gamma(a)} x^{-a} - v$$

These concepts were successfully applied by Liouville to hypothetical theoretical issues. The second technique cannot be used for a large class of FCS, whereas the first definition is limited to certain values of v.

THE FRACTIONAL DERIVATIVES FORMULA

The following provides proof for the FD formula:

$$D_x^\mu\left[x^\alpha(x-1)^\beta\left(1-\frac{w_1}{x-1}\right)^{-\gamma}\left(1-\frac{w_2}{x-1}\right)^{-\delta}\right]$$
$$=A\cdot F_{ci}\left[\begin{matrix}-\beta,-\beta,-\beta,-\mu,\gamma,\delta;1+\alpha-\mu,-\beta,-\beta;\frac{x}{x-1},\\ \frac{w_1}{x-1},\frac{w_2}{x-1}\end{matrix}\right],$$
$$\left|\frac{w_1}{x-1}\right|<1,|xw_2|<1,$$
$$D_x^\mu\left[x^\alpha(x-1)^\beta(1-xw_1)^{-\gamma}(1-xw_2)^{-\delta}\right]$$
$$=AF_s\left[\begin{matrix}-\beta,1+\alpha,1+\alpha,-\mu,\gamma,\delta;1+\alpha-\mu,1+\alpha-\mu,1+\alpha-\mu;\\ \frac{x}{x-1},xw_1,xw_2\end{matrix}\right],$$

where

$$\left|\frac{x(w_1+w_2)}{x-1}\right|<1$$
$$D_x^\alpha\left[x^\alpha(x-1)^\beta\left(1-\frac{w_1}{x-1}-\frac{w_2}{x-1}\right)^{-\gamma}\right]$$
$$D_x^\mu\left[x^a(x-1)^a(1-xw_1-xw_2)^{-r}\right]$$
$$A=x^{\alpha-\mu}(x-1)^\beta e^{-\lambda\pi\mu}\frac{\Gamma(\mu-\alpha)}{\Gamma(-\alpha)},$$
$$B=x^{\alpha-\mu}(x-1)^{-\gamma}e^{-iz\mu}\frac{\Gamma(\mu-\alpha)}{\Gamma(-\alpha)}. \qquad (3)$$

The aim is to develop efficient and reliable approximate methods that can yield accurate solutions over a wide range of parameter values. The FCS FQ, FN, and Fs are respectively defined above by as the Saran's FCS of three VB, and $F'[x,y,z]$ are the triple HGS defined by (3).

$$\left(z^\beta\right)_\alpha=e^{-i\pi\alpha}\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}z^{\beta-\alpha},\left|\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}\right|<\infty$$
$$\left((z-a)^\beta\right)_\alpha=e^{-i\pi\alpha}\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}(z-a)^{\beta-\alpha},\left|\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}\right|<\infty$$
$$(u.v)_\alpha=\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+1)}{\Gamma(1+\alpha-n)\Gamma(n+1)}u_{\alpha-n}v_n.$$

Proof:
We know that

$$(1-x)^{-\gamma}=\sum_{n=0}^{\infty}\frac{(\gamma)^n}{n!}x^n,|x|<1$$

The LHS of (2) is given by using (3) since the sequence of differentiation and summation is interchangeable under the aforementioned circumstances.

$$=\sum_{k,m+,H=0}^{\infty}\frac{(\gamma)_m(\delta)_n w_1^{m'}w_2^n}{m!\,n!}\frac{\Gamma(\mu+1)}{\Gamma(\mu+1-k)\Gamma(k+1)}(x^\alpha)_{\mu-k}\left((x-1)^{\beta-m-\mu}\right)_k$$
$$=\sum_{k,n+,n=0}^{\infty}\frac{(\gamma)_m(\delta)_n w_1^{m!}w_2^n}{k!\,m!\,n!}\frac{\Gamma(\mu+1)}{\Gamma(\mu+1-k)}\frac{\Gamma(\mu-k-\alpha)}{\Gamma(-\alpha)}\frac{\Gamma(k+m+n-\beta)}{\Gamma(m+n-\beta)}$$
$$\times e^{-i\pi\mu}x^{\alpha-\mu+k}(x-1)^{\beta-m-n-k}$$
$$=e^{-i\pi\mu}\frac{\Gamma(\mu-\alpha)}{1^-(-\alpha)}x^{\alpha-\mu}(x-1)^\beta$$
$$\times\sum_{k,m,n=0}^{\infty}\frac{(-\beta)_{k+m+n}(-\mu)_k(\gamma)_m(\delta)_n}{(1+\alpha-\mu)_k(-\beta)_{m+1}k!\,m!\,n!}\left(\frac{x}{x-1}\right)^k\left(\frac{w_1}{x-1}\right)^{m'}\left(\frac{w_2}{x-1}\right)^n$$

The generalization of (1) and (3) may be produced in the following way, it is crucial to note:

$$D_x^{ax}\left[x^{\alpha}(x-1)^{\beta}\left(1-\frac{w_1}{x-1}\right)^{-\gamma_1}\left(1-\frac{w_2}{x-1}\right)^{-\gamma_2}\cdots\left(1-\frac{w_n}{x-1}\right)^{-\gamma_n}\right]$$

$$= A_{(1)}^{(1)}E_{lj}^{(n+1)}\left[-\beta,-\mu,\gamma_1,\cdots,\gamma_n;1+\alpha-\mu,-\beta;\frac{x}{x-1},\frac{w_1}{x-1},\cdots,\frac{w_n}{x-1}\right]$$

And

$$D_x^{\mu}\left[x^{\alpha}(x-1)^{\beta}(1-xw_1)^{-\gamma_1}\cdots\cdots(1-xw_n)^{-\gamma_1}\right]$$

$$= A_{(2)}^{(1)}E_D^{(n+1)}\left[\begin{matrix}-\beta,1+\alpha,-\mu,\gamma_1,\cdots,\gamma_n;1+\alpha-\mu;\frac{x}{x-1},\\ xw_1,\ldots\ldots xw_n\end{matrix}\right],\cdots. \quad (5)$$

where $\frac{k}{1}\mathrm{E}\frac{n}{d}$ and $\frac{k}{2}\mathrm{E}\frac{n}{d}$ are the multiple HGS defined by (5).

LINEAR, DOUBLE AND MULTIPLE GF

We take into account the following fundamental identities.

$$[(1-x)-t]^{-\lambda}=(1-t)^{-\lambda}\left(1-\frac{x}{1-t}\right)^{-\lambda}$$

And

$$[1-(1-x)t]^{-\lambda}=(1-t)^{-\lambda}\left(1+\frac{xt}{1-t}\right)^{-\lambda}$$

Now, let us write

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}(1-x)^{-(\lambda+11)}t^n=(1-t)^{-\lambda}\left[1-\frac{x}{1-t}\right]^{-\lambda},|t|<|1-x|.$$

Replace x by $\frac{x(w1+w2)}{x-1}$ multiply both sides of $\mathrm{X}^{\alpha}(\mathrm{x}-1)^{-\beta}$ multiply both sides of (4.3) by $D\frac{\mu}{x}$ we get

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}t^nD_x^{\mu}\left[x^{\alpha}(x-1)^{-\beta}\left(1-\frac{x(w_1+w_2)}{x-1}\right)^{-(\lambda+n)}\right]$$

$$=(1-t)^{-\lambda}D_x^{\mu}\left[x^{\alpha}(x-1)^{-\beta}\left(1-\frac{x(w_1+w_2)}{(x-1)(1-t)}\right)^{-i}\right]$$

In order to derive the following two linear GF, we apply the approach used to derive (5) and utilize outcomes (5) and (4), respectively.

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{(3)}\left[\begin{matrix}-\beta::\lambda+n;-\mu;w_1\\ -\beta;1+\alpha-\mu;\frac{w_2}{x-1},\frac{x}{x-1}\end{matrix}\right]t^n$$

$$=(1-t)^{-\lambda}F^{(3)}\left[-\beta::\lambda;-\mu;\frac{w_1}{},\frac{w_2}{(x-1)(1-t)},\frac{x}{x-1}\right]$$

Now, replace x by $\frac{w_1}{x-1},\frac{w_2}{x-1}$ respectively, replace t by $\mathrm{t_1t_2}$ and λ by $\lambda_1\lambda_2$ respectively in (3). The two EQT are then multiplied by one another to produce

$$=\sum_{m,n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n}{m!\,n!}t_1^{m)}t_2^n\left(1-\frac{w_1}{x-1}\right)^{-(\lambda_1+m)}\left(1-\frac{w_2}{x-1}\right)^{-(\lambda_2+n)}$$

$$=(1-t)^{-\lambda_1}(1-t_2)^{-\lambda_2}\left(1-\frac{w_1}{(x-1)(1-t_1)}\right)^{-\lambda_1}\left(1-\frac{w_2}{(x-1)(1-t_2)}\right)^{-\lambda_2}$$

Mathematicians like Leonhard Euler, who studied fractional derivatives in the 18th century, are credited with developing fractional calculus. Multiply both sides of (3) by $\mathrm{x}^{\alpha}(\mathrm{x}-1)^{\beta}$ and then use the FD operator to operate both sides and using (2), the subsequent double GF is what we get:

$$\sum_{m,n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n}{m!n!}t_1^{m!}t_2^{n} F_G\left[\begin{matrix}-\beta,-\mu,\lambda_1+m,\lambda_2+n;1+\alpha-\mu,-\beta;\frac{x}{x-1},\\ \frac{w_1'}{x-1},\frac{w_2}{x-1}\end{matrix}\right] \quad (6)$$

$$=(1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_1} F_G\left[\begin{matrix}-\beta,-\mu,\lambda_1,\lambda_2;1+\alpha-\mu,-\beta;\frac{x}{x-1},\\ \frac{w_1}{(x-1)(1-t_1)},\frac{w_2}{(x-1)(1-t_2)}\end{matrix}\right].$$

The following is how the GF's generalization (4.6) may be obtained:

$$\sum_{m_1,\cdots,m_n=0}^{\infty}\frac{(\lambda_1)_{m_1}\cdots\cdots(\lambda_n)_{m_n}}{m_1!\cdots\cdots m_n!}t_1^{m_1}\cdots\cdots t_n^{m_n} \times_{(1)}^{(1)}E_D^{(n+1)}\left[\begin{matrix}-\beta,-\mu,\lambda_1+m_1,\cdots\cdots,\lambda_n+m_n;\\ 1+\alpha-\mu,-\beta;\frac{x}{x-1},\frac{w_1}{x-1},\cdots\cdots,\frac{w_n}{x-1}\end{matrix}\right] \quad (7)$$

$$=\quad(1-t_1)^{-\lambda_1}\cdots\cdots(1-t_n)^{-\lambda_1} x_{(1)}^{(1)}E_{b)}^{(n+1)}\left[\begin{matrix}-\beta,-\mu,\lambda_1,\cdots,\lambda_n;1+\alpha-\mu,-\beta;\frac{x}{x-1},\\ \frac{w_t}{(x-1)(1-t_1)},\cdots,\frac{w_n}{(x-1)(1-t_n)}\end{matrix}\right].$$

As a consequence of adopting the methodology used to arrive at (7) and this result can be obtained on the similar lines of the proof of the theorem using the outcomes from (6) and (7), following double GF are then obtained:

$$\sum_{m,n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n}{m!n!}t_1^{m'}t_2^{n} F_N\left[\begin{matrix}\lambda_1+m,\lambda_2+n,-\mu,-\beta,1+\alpha,-\beta;-\beta,1+\alpha-\mu,\\ 1+\alpha-\mu,\frac{w_1}{x-1},xw_2\frac{x}{x-1}\end{matrix}\right] \quad (8)$$

$$=(1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2} F_N\left[\begin{matrix}\lambda_1,\lambda_2,-\mu,-\beta,1+\alpha,-\beta;-\beta,1+\alpha-\mu,1+\alpha-\mu;\\ \frac{w_1}{(x-1)(1-t_1)},\frac{xw_2}{1-t_2},\frac{x}{x-1}\end{matrix}\right]$$

And

$$\sum_{m,n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n}{m!\,n!}t_1^{m}t_2^{n} F_s\left[\begin{matrix}-\beta,1+\alpha,1+\alpha,-\mu,\lambda_1+m,\lambda_2+n;1+\alpha-\mu,\\ 1+\alpha-\mu,1+\alpha-\mu;\end{matrix}\right.$$

$$\left.=\frac{x}{x-1},xw_1,xw_2\right]$$

$$=(1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}F_s\left[\begin{matrix}-\beta,1+\alpha,1+\alpha,-\mu,\lambda_1,\lambda_2;\\ 1+\alpha-\mu,1+\alpha-\mu,1+\alpha-\mu;\end{matrix}\right.$$

$$\left.\frac{x}{x-1},\frac{xw_1}{1-t},\frac{xw_2}{1-t}\right].$$

$$\sum_{m_1\ldots m_n-0}^{\infty}\frac{(\lambda_1)_{m_1}\cdots(\lambda_n)_{m_n}}{m_1!\ldots m_n!}t_1^{m_1}\ldots t_n^{m_n} {}_{(2)}^{(1)}E_l^{(n+1)}\left[\begin{matrix}-\beta,1+\alpha,-\mu,\lambda_1+m_1,\ldots\ldots,\lambda_n+m_n;1+\alpha-\mu;\\ \frac{x}{x-1},xw_1,\ldots,xw_n\end{matrix}\right] \quad (9)$$

$$=(1-t_1)^{-\lambda_1}\ldots(1-t_n)^{-\lambda_2} (1)\,E_D^{(n+1)}\left[\begin{matrix}-\beta,1+\alpha,-\mu,\lambda_1,\ldots,\lambda_n;1+\alpha-\mu;\frac{x}{x-1},\\ \frac{xw_1}{1-t},\cdots,\frac{xw_n}{1-t}\end{matrix}\right].$$

Now, we establish some additional GF using the identity (9). Let's express the identity (9) as:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}(1-x)^n t^n = (1-t)^{-\lambda}\left(1-\frac{xt}{t-1}\right)^{-\lambda}, |t| < |1-x|^{-1} (10)$$

Multiply both sides of (10) by $(1-x)^{-\rho}$ and replacing x by $\frac{x(w_1+w_2)}{x-1}$, we have

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\left(1-\frac{x(w_1+w_2)}{x-1}\right)^{n-\rho} t^n = (1-t)^{-\lambda}\left(1-\frac{x(w_1+w_2)}{x-1}\right)^{-\rho}\left(1-\frac{tx(w_1+w_2)}{(x-1)(t-1)}\right)^{-\lambda} (11)$$

Now, multiply (11), on both sides, by $x^a(x-1)^{-(\alpha+1)}$, operate by FD operator $D\frac{\mu}{x}$ and using (4), then, we arrive to the subsequent GF:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F^{(3)}.\left[\begin{matrix}\alpha+1 & :: & \rho-n;-\mu;xw_1 \\ 1+\alpha-\mu:: & & \frac{xw_2}{x-1},\frac{x}{x-1}\end{matrix}\right] t^n \quad (12)$$

$$= (1-t)^{-\lambda} F_b^{(3)}\left[1+\alpha,-\mu,\rho,\lambda;1+\alpha-\mu;\frac{x(w_1+w_2)}{x-1},\frac{xt(w_1+w_2)}{(x-1)(t-1)}\right].$$

$$= (1-t)^{-\lambda} F_G\left[-\beta,-\mu,\rho,\lambda;1+\alpha-\mu,-\beta \; \frac{x}{x-1},\frac{w_1+w_2}{x-1},\frac{t(w_1+w_2)}{(x-1)(t-1)}\right] (4.13)$$

$$= (1-t)^{-\lambda} F_s\left[-\beta,1+\alpha,-\mu,\rho,\lambda;1+\alpha-\mu \; \frac{x}{x-1},x(w_1+w_2),\frac{xt(w_1+w_2)}{t-1}\right]$$

BILINEAR, DOUBLE AND MULTIPLE GF

$$[(1-x)(1-y)-t]^{-\lambda} = (1-y)^{-\lambda}\left[\left(1-\frac{x}{1-t}\right)\left(1-\frac{y}{1-t}\right)-\frac{xyt}{(1-t)^2}\right]^{-\lambda} \dots\dots$$

write (4) as

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}(1-x)^{-(\lambda+n)}(1-y)^{-(\lambda+n)}t^n$$

$$= (1-t)^{-\lambda}\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\left(1-\frac{x}{1-t}\right)^{-(\lambda+n)}\left(1-\frac{y}{1-t}\right)^{-(\lambda+n)}\left(\frac{xyt}{(1-t)^2}\right)^n.$$

Again replace x, y, t and A, by $w\_2/(x-1), z\_2/(y-1)$ t_2 and λ_2 respectively.

$$\sum_{n!n=0}^{\infty} \frac{(\lambda_1)_{n!}(\lambda_2)_n}{m!n!}\left(1-\frac{w_1}{x-1}\right)^{-(\lambda_1+m)}\left(1-\frac{w_2}{x-1}\right)^{-(\lambda_2+n)}$$

$$\times\left(1-\frac{z_1}{y-1}\right)^{-(\lambda_1+\pi)}\left(1-\frac{z_2}{y-1}\right)^{-(\lambda_2+n)} t_1^m t_2''$$

$$= (1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}\sum_{m,n=0}^{\infty} \frac{(\lambda_1)_m(\lambda_2)_n}{m!n!} \quad (13)$$

$$\left(\frac{w_1 z_1 t_1}{(x-1)(y-1)(1-t_1)^2}\right)^{nt}\left(1-\frac{w_1}{(x-1)(1-t_1)}\right)^{-(\lambda_1+n+1}$$

$$\times\left(1-\frac{w_2}{(x-1)(1-t_2)}\right)^{-(\lambda_2+1)}\left(1-\frac{z_2}{(y-1)(1-t_2)}\right)^{-(\lambda_2+n)}$$

Now, multiply both sides of (3) by $x^\alpha(x-1)^\beta y^y(y-1)^{\mathcal{L}}$. Then operate both sides by FD operators $D\frac{\mu}{x}$ and $D\frac{y}{y}$ respectively and using (1), we arrive at the following double GF:

$$\sum_{m,n=0}^{\infty} \frac{(\lambda_1)_m(\lambda_2)_n}{m!\,n!} t_1^{n!} t_2^{n}$$

$$F_G\begin{bmatrix} -\beta,-\mu,\lambda_1+m,\lambda_2+n;1+\alpha-\mu,-\beta; \\ \frac{x}{x-1},\frac{w_1}{x-1},\frac{w_2}{x-1} \end{bmatrix}$$

$$F_{ij}\begin{bmatrix} -\delta,-v,\lambda_1+m,\lambda_2+n;1+\gamma-v,-\delta; \\ -\frac{y}{y-1},\frac{z_1}{y-1},\frac{z_2}{y-1} \end{bmatrix}$$

$$=(1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}$$

$$\sum_{n+,n=0}^{\infty} \frac{(\lambda_1)_{nn}(\lambda_2)_n}{m!\,n!}\left(\frac{w_1 z_1 t_1}{(x-1)(y-1)(1-t_1)^2}\right)^{m+}\left(\frac{w_2 z_2 t_2}{(x-1)(y-1)(1-t_2)^2}\right)^{n}$$

$$F_\sigma\begin{bmatrix} m+n-\beta,m+n-\beta,m+n-\beta,-\mu, \\ \lambda_1+m,\lambda_2+n \\ 1+\alpha-\mu,m+n-\beta,m+n-\beta;\frac{x}{x-1}, \\ \frac{w_1}{(x-1)(1-t_1)},\frac{w_2}{(x-1)(1-t_2)} \end{bmatrix}$$

$$F_G\begin{bmatrix} m+n-\delta,m+n-\delta,m+n-\delta,-v, \\ \lambda_1+m,\lambda_2+n \\ 1+\gamma-v,m+n-\delta,m+n-\delta;\frac{y}{y-1}, \\ \frac{z_1}{(y-1)(1-t_1)},\frac{z_2}{(y-1)(1-t_2)} \end{bmatrix}. \quad (14)$$

Generalized polynomial is actually introduced by Srivastava of the following manner: the following form can be used to derive the generalization of (14):

$$\sum_{m_1,\cdots,m_n<0}^{\infty} \frac{(\lambda_1)_{m_1}\cdots\cdots(\lambda_n)_{m_m}}{m_1!\cdots\cdots m_n!} t_1^{m_1}\cdots\cdots t_n^{n_n}$$

$$(1)\,E_{b)}^{(n+1)}\begin{bmatrix} -\beta,-\mu,\lambda_1+m_1,\cdots,\lambda_n+m_n;1+\alpha-\mu,-\beta; \\ \frac{x}{x-1},\frac{w_1}{x-1},\cdots,\frac{w_n}{x-1} \end{bmatrix}$$

$$(1)\,E_{p}^{(n+1)}\begin{bmatrix} -\delta,-v,\lambda_1+m_1,\cdots\cdots,\lambda_n+m_n;1+\gamma-v,-\delta; \\ \frac{y}{y-1},\frac{z_1}{y-1},\cdots,\frac{z_n}{y-1} \end{bmatrix}$$

$$=(1-t_1)^{-\lambda_1}\cdots(1-t_n)^{-\lambda_n}\cdots\left(\frac{w_2 z_2 t_2}{(x-1)(y-1)(1-t_n)^2}\right)^{m_n}$$

$$(1)\,E_{D}^{(n+1)}\begin{bmatrix} m_1+\cdots+m_n-\beta,-\mu,\lambda_1+m_1,\cdots,\lambda_n+m_n; \\ 1+\alpha-\mu,m_1+\cdots+m_n-\beta \\ \frac{x}{x-1},\frac{w_1}{(x-1)(1-t_1)},\cdots,\frac{w_n}{(x-1)(1-t_n)} \end{bmatrix};$$

$$(1)\,E_{D}^{(n+1)}\begin{bmatrix} m_1+\cdots+m_n-\delta,-v,\lambda_1+m_1,\cdots,\lambda_n+m_n; \\ 1+\gamma-v,m_1+\cdots+m_n-\delta;\frac{y}{y-1},\frac{z_1}{(y-1)(1-t_1)},\cdots, \\ \frac{z_n}{(y-1)(1-t_n)} \end{bmatrix}$$

We now apply the technique used to generate (14) and use the findings (2) and (3), arriving at the And

$$\sum_{m\cdot n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n t_1^{m''}t_2''}{m!\,n!}$$

$$F_S\left[\begin{matrix}-\beta,1+\alpha,1+\alpha,-\mu,\lambda_1+m,\lambda_2+n;1+\alpha-\mu,\\ 1+\alpha-\mu,1+\alpha-\mu;\frac{x}{x-1},xw_1,xw_2\end{matrix}\right]$$

$$F_s[-\delta,1+\gamma,1+\gamma,-v,\lambda_1+m,\lambda_2+n;1+\gamma-v,1+\gamma-v,1+\gamma-v;$$

$$=(1-t_1)^{-\lambda_1}(1-t_1)^{-\lambda_2}\sum_{m,n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n}{m!\,n!}$$

$$\left(\frac{xyw_1z_1t_1}{(1-t_1)^2}\right)^m\left(\frac{xyw_2z_2t_2}{(1-t_2)^2}\right)^n\frac{(1+\alpha)_{m+n}(1+\gamma)_{n+n},yz_1,yz_2}{(1+\alpha-\mu)_{m+n}(1+\gamma-v)_{m+n}}\Bigg]$$

$$F_s[-\beta,1+\alpha+m+n,1+\alpha+m+n,-\mu,\lambda_1+m,\lambda_2+n;$$
$$1+\alpha+m+n-\mu,1+\alpha+m+n-\mu,1+\alpha+m+n-\mu;\frac{x}{x-1},\frac{xw_1}{1-t_1},\frac{xw_2}{1-t_2}\Bigg]$$

$$F_s[-\delta,1+\gamma+m+n,1+\gamma+m+n,-v,\lambda_1+m,\lambda_2+n;$$
$$1+\gamma+m+n-v,1+\gamma+m+n-v,1+\gamma+m+n-v;\frac{y}{y-1},\frac{yz_1}{1-t_1},\frac{yz_2}{1-t_2}\Bigg]$$

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$$\sum_{m_n\ldots m_n=0}^{\infty}\frac{(\lambda_1)_{m_1}\ldots\ldots(\lambda_n)_{m_n}}{m_1!\ldots m_n!}t_1^{m_1}\ldots\ldots t_n^{m_n}$$

$${}^{(2)}_{(2)}E_D^{(n+1)}\left[\begin{matrix}-\beta,1+\alpha,-\mu,\lambda_1+m_1,\ldots.,\lambda_n+m_\mu;\\ 1+\alpha-\mu;\frac{x}{x-1},xw_1,\ldots,,xw_n\end{matrix}\right]$$

$${}^{(1)}_{(2)}E_{1\rangle}^{(n+1)}\left[\begin{matrix}-\delta,1+\gamma,-v,\lambda_1+m_n,\ldots\ldots,\lambda_n+m_n;\\ 1+\gamma-v;\frac{y}{y-1},yz_1,\ldots.,yz_n\end{matrix}\right]$$

$$=(1-t_1)^{-\lambda_1}\ldots(1-t_n)^{-\lambda}\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\lambda_1)_{n_1}\ldots\ldots(\lambda_n)_{nN_n}}{m_1!\ldots m_n!}$$

$$\left(\frac{xyw_1z_1t_1}{(1-t_1)^2}\right)^{m_1}\ldots\left(\frac{xyw_nz_nt_n}{(1-t_n)^2}\right)^{m_n}\frac{(1+\alpha)_{m_1+\cdots+m_r}(1+\gamma)_{m_1+\cdots+m_n}}{(1+\alpha-\mu)_{m_1+\cdots+m_n}(1+\gamma-v)_{m_1+\cdots+w_n}}$$

$$(2)\ E_{D)}^{(n+1)}[-\beta,1+\alpha+m_1+\cdots..+m_n,-\mu,\lambda_1+m_1,\ldots\ldots,\lambda_n+m_n;$$

$$1+\alpha+m_1+\cdots.+m_n-\mu;\frac{x}{x-1},\frac{xw_1}{1-t_1},\ldots,\frac{xw_n}{1-t_n}\Bigg]$$

$${}^{(1)}_{(2)}E_D^{(n+1)}[-\delta,1+\gamma+m_1+\cdots.+m_n,-v,\lambda_1+m_1,\ldots,\lambda_n+m_n;$$
$$1+\gamma+m_1+\cdots+m_n-v;\frac{y}{y-1},\frac{yz_1}{1-t_1},\ldots,\frac{yz_n}{1-t_n}\Bigg] \quad (15)$$

Now, in (4), we replace x and y by $x(w\_1+w\_2)/(x-1)$ and $y(z\_1+z\_2)/(y-1)$ respectively and then multiply both of it by $x^\alpha y^\beta\{x-1)^{-\gamma}(y-1)^{\mathcal{A}}$ and then operate by $D\frac{\mu}{x}$ and $D\frac{v}{y}$ for x and y respectively and using (2), we obtain the following bilinear GF:

$$= (1-t)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \left(\frac{xyt(w_1+w_2)(z_1+z_2)}{(x-1)(y-1)(1-t)^2} \right)^n \frac{(1+\alpha)_n(1+\beta)_n}{(1+\alpha-\mu)_n(1+\beta-v)_n}$$

$$F^{(3)} \left[\begin{matrix} \gamma+n:\lambda+n, 1+\alpha+n; -; -: -; -; -\mu; \\ 1+\alpha+n-\mu :: \gamma+n ; -; \frac{xw_1}{(x-1)(1-t)}, \frac{xw_2}{(x-1)(1-t)}, \frac{x}{x-1} \end{matrix} \right]$$

Using the aid of the outcomes from (5) and (9), we used the same procedure to acquire the similarly intriguing bilinear GF that are shown below

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F^{(3)} \left[\begin{matrix} -\gamma: \lambda+n; \ -\mu; w_1 \\ -\gamma; 1+\alpha-\mu; \frac{w_2}{x-1}, \frac{x}{x-1} \end{matrix} \right]$$

$$= (1-t)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \left(\frac{(w_1+w_2)(z_1+z_2)t}{(1-t)^2(x-1)(y-1)} \right)^n$$

$$F^{(3)} \left[n-\gamma :: \lambda+n; -\mu; \frac{w_1}{-:: n-\gamma; ; -; 1+\alpha-\mu;}; , \frac{w_2}{(x-1)(1-t)}, \frac{x}{x-1} \right]$$

$$F^{(3)} \left[n-\delta: : \lambda+n; \ -v; \frac{z_1}{-}, \frac{z_2}{-:: n-\delta; -; 1+\beta-v;}, \frac{y}{y-1} \right]$$

And

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F^{(3)} \left[\begin{matrix} - \quad : \lambda+n, 1+\alpha; -\mu, -\gamma; ; \\ 1+\alpha-\mu \quad ;; \\ xw_2, \frac{x}{x-1} \end{matrix} \right]$$

$$= (1-t)^{-i} \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \left(\frac{(w_1+w_2)(z_1+z_2)xyt}{(1-t)^2} \right)^n \frac{(1+\alpha)_n(1+\beta)_n}{(1+\alpha-\mu)_n(1+\beta-v)_n}$$

BA GF

In this part, we build some BA GF using the linear GF [described in section. In equation (5), we change t to $t(1-y)$, multiply both sides by y^γ and operate by $D\frac{v}{y}$ (for the variable y), we obtain

$$y^\gamma[1-t(1-y)]^{-\lambda})$$

A significant amount of theoretical work has been done in the topic of FC. We now arrive at the following BA GF using (7)-(9) and some standard calculations:

$${}_2F_1[-n, 1+\gamma; 1+\gamma-v; y]t^n$$

$$= (1-t)^{-\lambda} \textstyle\sum_{p=0}^{\infty} \frac{(\lambda)_p(1+\alpha)_p}{(1+\alpha-\mu)_p p!} \left(\frac{xw_1}{(x-1)(1-t)} \right)^p (4.16)$$

the following two BA GF are obtained:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F^{(3)} \left[\begin{matrix} -\beta: \lambda+n; -\mu; \\ -: -\beta; 1+\alpha-\mu; \frac{w_1}{x-1}, \frac{w_2}{x-1}, \frac{x}{x-1} \end{matrix} \right]$$

$${}_2F_1[-n, 1+\gamma; 1+\gamma-v; y]t''$$

$$= (1-t)^{-\lambda} \sum_{p=0}^{3} \frac{(\lambda)_p}{p!} \left(\frac{n_1}{(x-1)(1-t)} \right)^p$$

$$\left. \frac{-yt}{1-t}, \frac{x}{x-1}, \frac{w_2}{(x-1)(1-t)} \right]$$

ultiply both sides by y^y and then operate by $D\frac{v}{y}$ (for the variable y), we obtain

$$D_r^v\left(\sum_{m,n=1}^{\infty}\frac{(\lambda_1)_{nt}(\lambda_2)_n}{m!\,n!}t_1^{m'}t_2^{\prime\prime}(1-\eta_1 y)^{n'}(1-\eta_2 y)^n \times F_c\left[\begin{matrix}-\beta,-\mu,\lambda_1+m,\lambda_2+n;1+\alpha-\mu,-\beta;\frac{x}{x-1},\\ \frac{w_1}{x-1},\frac{w_2}{x-1}\end{matrix}\right]\right)$$

$$= D_5^r\left(\left[y^{\gamma}[1-t_1(1-\eta_1 y)]^{-\lambda_1}[1-t_2(1-\eta_2 y)]^{-\lambda_2}\right] \times F_{ij}\left[\begin{matrix}-\beta,-\mu,\lambda_1,\lambda_2;1+\alpha-\mu,-\beta;\frac{x}{x-1},\\ \frac{w_1}{(x-1)(1-t_1)},\frac{w_2}{(x-1)(1-t_2)}\end{matrix}\right]\right)$$

We now arrive at the following BA GF using (7)-(9) and some standard calculations:

$$\sum_{n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n}{m!\,n!}t_1^{\prime\prime\prime}t_2^n$$

$$F_G\left[\begin{matrix}-\beta,-\mu,\lambda_1+m,\lambda_2+n;1+\alpha-\mu,-\beta;\\ \frac{x}{x-1},\frac{w_1}{x-1},\frac{w_2}{x-1}\end{matrix}\right]$$

$$F_1[1+\gamma,-m,-n;1+\gamma-v;\eta_1 y,\eta_2 y]$$

$$=(1-t_1)^{-\lambda_1}\sum_{4,r=0}^{\infty}\frac{(\lambda_1)_q(\lambda_2)_r}{q!\,r!}\left(\frac{w_1}{(x-1)(1-t_1)}\right)^q\left(\frac{w_2}{(x-1)(1-t_2)}\right)^q$$

$${}_2F_1\left[-\beta+q+r_1-\mu;1+\alpha-\mu;\frac{x}{x-1}\right]$$

$$F_1\left[1+\gamma,\lambda_t+q,\lambda_2+r;1+\gamma-v;\frac{t_1\eta_1 y}{t_1-1},\frac{t_2\eta_2 y}{t_2-1}\right].$$

Further, if in (9), we replace t_1 and t_2 by $t_1(1-\eta_1,y)$ and $t_2(1-\eta_2,y)$) respectively, such that $|\eta_1|<1$, i = 1,2.

$$\sum_{n,n=0}^{\infty}\frac{(\lambda_1)_n(\lambda_2)_n}{m!\,n!}t_1^{n'}t_2^n$$

$$F_N\left[\begin{matrix}\lambda_1+m,\lambda_2+n,-\mu,-\beta,1+\alpha,-\beta;-\beta,1+\alpha-\mu,1+\alpha-\mu;\frac{w_1}{x-1},\\ xw_2,\frac{x}{x-1}\end{matrix}\right]$$

$$F_1[1+\gamma,-m,-n;1+\gamma-v;\eta_1 y,\eta_2 y]$$

$$=(1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}\sum_{p,4=0}^{\infty}\frac{(\lambda_1)_p(\lambda_2)_q(1+\alpha)_4}{(1+\alpha-\mu)_4\,p!\,q!}\left(\frac{w_2}{(x-1)(1-t_2)}\right)^4$$

$${}_2F_1\left[-\beta+p,-\mu;1+\alpha-\mu;\frac{x}{x-1}\right]\quad \ldots\ldots(4.5.7)$$

$$F_1\left[1+\gamma,\lambda_1+p,\lambda_2+q;1+\gamma-v;\frac{t_1}{t_1-1},\frac{\eta_1 y}{t_2-1}\right]$$

$$\sum_{x_0,n=0}^{\infty} \frac{(\lambda_1)_m (\lambda_2)_n}{m!\,n!} t_1^{m'} t_2^m$$

$$F_s \begin{bmatrix} -\beta, 1+\alpha, 1+\alpha, -\mu, \lambda_1+m, \lambda_2+n; 1+\alpha-\mu, 1+\alpha-\mu, \\ 1+\alpha-\mu; \frac{x}{x-1}, xw_1, xw_2 \end{bmatrix}$$

$$F_1[1+\gamma, -m, -n; 1+\gamma-v; \eta_1 y, \eta_2 y]$$

$$= (1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2} \sum_{q,r=0}^{m} \frac{(\lambda_1)_q(\lambda_2)_r(1+\alpha)_{4+r}}{(1+\alpha+\mu)_{4+r}\,q!\,r!} \left(\frac{xw_1}{1-t_1}\right)^{\mu} \left(\frac{xw_2}{1-t_2}\right)^{r}$$

$${}_2F_1\left[-\beta, -\mu; 1+\alpha-\mu; \frac{x}{x-1}\right]$$

$$F_1\left[1+\gamma, \lambda_1+q, \lambda_2+r; 1+\gamma-v; \frac{t_1\eta_1 y}{t_1-1}, \frac{t_2\eta_2 y}{t_2-1}\right].$$

2. Conclusion:

In this section, we apply the concept of Nishimoto's FC (NFC) to show how to calculate the FD of triple HGFs in three dimensions. We will utilize these EQT to derive GF for linear, bilinear, and BA sequences as part of our study. Based on the integral representations for quadruple hypergeometric functions (1)–(8), we obtained certain generating functions for these functions. Some particular cases and the consequences of our main results are also considered. We concluded this investigation by remarking that the scheme suggested in the derivation of the results can be applied to find other new generating functions for other quadruple hypergeometric functions and study their special cases.

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