PROBABILITY DISTRIBUTION OF INTEGRAL INVOLVING HYPERGEOMETRIC FUNCTION

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4.1 MAIN INTEGRALS

In this section, the following probability distribution of thirty nine integrals involving hypergeometric functions have been obtainedin the form of a single integral.

$$\int_0^1 x^{c-1}(1-x)^{c-e+1}[.1+\alpha x+\beta(1-x)]^{-2c+e-i-1}F_1^2\left(a,1+j-a',e:\frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)}\right)dx =$$

$$\frac{\Gamma(e)\Gamma(c-e+i+1)\Gamma(e-c-\frac{1}{2}(i+|i|))\Gamma(c-\frac{1}{2}(j+|j|))\Gamma(a-\frac{1}{2}(i+j+|i+j|))}{2^{2a-i-j}(1+\alpha)^{c}(1+\beta)^{c-e+i+1}\Gamma(e-a)\Gamma(e-c)\Gamma(a)\Gamma 2c-e-a+i+1)}\text{x}\{D_{i,j}$$

$$\frac{\Gamma(c+\frac{1}{2i}-\frac{1}{2e}-\frac{1}{2a}+\frac{1}{2})\Gamma(\frac{1}{2e}-\frac{1}{2a}+\frac{1}{4}(1-+(-1)^{i}))}{\Gamma(c-\frac{1}{2e}+\frac{1}{2a}+\frac{1}{2}+\frac{(-1)^{i}}{4})((-1)^{i}-1+[-j/2])\Gamma(\frac{1}{2e}+\frac{1}{2a}-\frac{1}{2i}+\left[-\frac{j}{2}\right])}+$$

$$E_{i,j}\frac{\Gamma(c+\frac{1}{2i}-\frac{1}{2e}-\frac{1}{2a}+1)\Gamma(\frac{1}{2e}-\frac{1}{2a}+\frac{1}{4}(1+(-1)^{i}))}{\Gamma(c-\frac{1}{2e}+\frac{1}{2a}+\frac{1}{2}+\frac{(-1)^{j}}{4})((1-(-1)^{i})+[-\frac{j}{2}+\frac{1}{2}])\Gamma(\frac{1}{2e}+\frac{1}{2a}-\frac{1}{2i}-\frac{1}{2}+\left[-\frac{j}{2}+\frac{1}{2}\right])}\text{.........................}(4.1.1)$$

for i,j=0,±1, ±2, ±3.

Also, provided Re(e)>0,Re(c-e+i+1)>0 for i=0, ±1, ±2, ±3 and Re(c)>j for j=1,2,3.also the constants α and β are such that no one of the expressions 1+α,1+β and 1+αx+β(1-x), where 0≤ x ≤1, is zero. Again, as usual [x] is the greatest integer less than or equal to x. the coefficient $D_{i,j}$ and $E_{i,j}$ are given in the tabular form.

So by well known definition of probability distributions we have:

$$F(x)=\frac{\left\{\frac{\Gamma(e)\Gamma(c-e+i+1)\Gamma\left(e-c-\frac{1}{2}(i+|i|)\right)\Gamma\left(c-\frac{1}{2}(j+|j|)\right)\Gamma\left(a-\frac{1}{2}(i+j+|i+j|)\right)}{2^{2a-i-j}(1+\alpha)^{c}(1+\beta)^{c-e+i+1}\Gamma(e-a)\Gamma(e-c)\Gamma(a)\Gamma(2c-e-a+i+1)}\text{x}\left\{Di,j\frac{\Gamma\left(c+\frac{1}{2i}-\frac{1}{2e}-\frac{1}{2a}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2e}-\frac{1}{2a}+\frac{1}{4}(1\mp(-1)^{i})\right)}{\Gamma(c-\frac{1}{2e}+\frac{1}{2a}+\frac{1}{2}+\frac{(-1)^{i}}{4})((-1)^{i}-1+[-j/2])\Gamma(\frac{1}{2e}+\frac{1}{2a}-\frac{1}{2i}+\left[-\frac{j}{2}\right])}+Ei,j\frac{\Gamma(c+\frac{1}{2i}-\frac{1}{2e}-\frac{1}{2a}+1)\Gamma(\frac{1}{2e}-\frac{1}{2a}+\frac{1}{4}(1+(-1)^{i}))}{\Gamma(c-\frac{1}{2e}+\frac{1}{2a}+\frac{1}{2}+\frac{(-1)^{j}}{4})((1-(-1)^{i})+[-\frac{j}{2}+\frac{1}{2}])\Gamma(\frac{1}{2e}+\frac{1}{2a}-\frac{1}{2i}-\frac{1}{2}+\left[-\frac{j}{2}+\frac{1}{2}\right])}\right\}}{\int_0^1 x^{c-1}(1-x)^{c-e+i}[.1+\alpha x+\beta(1-x)]^{-2c+e-i-1}dx}$$

=0, elsewhere

=1, $\int_0^1 f(x)dx = 1$

Where f(x) = $F_1^2\left(a,1+j-a,e:\frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)}\right)$

4.2 SPECIAL CASES

1. If we set I,j=0, ±1, ±2, in (4.2.1),we get wenty five integrals obtained earlier by Nagar [108] and gaur(2003).

2. On the other hand, fourteen integrals for different values of I and j other than obtained by Nagar[108] and gaur(2003).

First Formula

$$\int_0^1 x^{c-1}(1-x)^{c-e+3}[.1+\alpha x+\beta(1-x)]^{-2c+e-4}F_1^2\left(a,1-a,e:\frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)}\right)dx =$$

$$\frac{\Gamma(e)\Gamma(c-e+4)\Gamma(e-c-3))\Gamma(c)}{2^{2a}(1+\alpha)^{c}(1+\beta)^{c-e+4}\Gamma(e-a)\Gamma(e-c)\Gamma(2c-e-a+4)}\text{x}[\{-(a+2)(a-3)+3c(c+3)-e(3c-e+5)\}\frac{\Gamma(c-\frac{a}{2}-\frac{e}{2}+2)\Gamma(\frac{e}{2}-\frac{a}{2}+\frac{1}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}-\frac{1}{2})\Gamma(c-\frac{e}{2}+\frac{a}{2}+2)}+\{(a+1)(a-2)-c(c+3)-e(c-e+3)\}\frac{\Gamma(c-\frac{e}{2}-\frac{a}{2}+\frac{5}{2})\Gamma(\frac{e}{2}-\frac{a}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2})\Gamma(c-\frac{e}{2}+\frac{a}{2}+\frac{3}{2})}\text{.........................}(4.2.1)$$

Provided Re(c)>0,Re(c-e+4)>0, Re(e)>0. Also the constants α and β are such that none of the expressions 1+α,1+β and 1+αx+β(1-x), 0≤ x ≤1, is not zero.
So by well known definition of probability distributions we have:

$$F(x)=\frac{\frac{\Gamma(e)\Gamma(c-e+4)\Gamma(e-c-3))\Gamma(c)}{2^{2a}(1+\alpha)^{c}(1+\beta)^{c-e+4}\Gamma(e-a)\Gamma(e-c)\Gamma(a)\Gamma 2c-e-a+4)}x\left[\{-(a+2)(a-3)+3c(c+3)-e(3c-e+5)\}\frac{\Gamma(c-\frac{a}{2}-\frac{e}{2}+2)\Gamma(\frac{e}{2}-\frac{a}{2}+\frac{1}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}-\frac{1}{2})\Gamma(c-\frac{e}{2}+\frac{a}{2}+2)}+\{(a+1)(a-2)-c(c+3)-e(c-e+3)\}\frac{\Gamma(c-\frac{e}{2}-\frac{a}{2}+\frac{5}{2})\Gamma(\frac{e}{2}-\frac{a}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2})\Gamma(c-\frac{e}{2}+\frac{a}{2}+\frac{3}{2})}\right.}{\int_0^1 x^{c-1}\,(1-x)^{c-e+i}[.1+\alpha x+\beta(1-x)]^{-2c+e-4}dx}$$

=0, elsewhere

$=1, \int_0^1 f(x)dx = 1$

Where f(x) = $F_1^2(a,1-a;\ e{:}\frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)})$

Second Formula

$$\int_0^1 x^{c-1}\,(1-x)^{c-e+3}[.1+\alpha x+\beta(1-x)]^{-2c+e-3}F_1^2(a,-a;\ e{:}\frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)})dx = \frac{\Gamma(e)\Gamma(c-e+3)\Gamma(e-c-2))\Gamma(c)}{2^{2a+1}(1+\alpha)^{c}(1+\beta)^{c-e+3}\Gamma(e-a)\Gamma(e-c)\Gamma(2c-e-a+3)}x[\{-(a+1)(a-2)+c(a+c+3)-e(2c-e+3)\}\frac{\Gamma(c-\frac{a}{2}-\frac{e}{2}+\frac{3}{2})\Gamma(\frac{e}{2}-\frac{a}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2})\Gamma(c-\frac{e}{2}+\frac{a}{2}+\frac{3}{2})}+\{(a-1)(a+2)+c(a-c+3)-e(2c-e+3)\}\frac{\Gamma(c+2-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2}+\frac{1}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}+\frac{1}{2})\Gamma(c-\frac{e}{2}+\frac{a}{2}+2)} \quad \text{..........................(4.2.2)}$$

Provided Re(c)>0,Re(c-e+4)>0, Re(e)>0. Also the constants α and β are such that none of the expressions 1+α,1+β and 1+αx+β(1-x), 0≤ x ≤1, is not zero.
So by well known definition of probability distributions we have:

$$F(x)=\frac{\frac{\Gamma(e)\Gamma(c-e+3)\Gamma(e-c-2))\Gamma(c)}{2^{2a+1}(1+\alpha)^{c}(1+\beta)^{c-e+3}\Gamma(e-a)\Gamma(e-c)\Gamma(a)\Gamma 2c-e-a+3)}x\left[\{-(a+1)(a-2)+c(a+c+3)-e(2c-e+3)\}\frac{\Gamma\left(c-\frac{a}{2}-\frac{e}{2}+\frac{3}{2}\right)\Gamma\left(\frac{e}{2}-\frac{a}{2}\right)}{\Gamma(\frac{e}{2}+\frac{a}{2})\Gamma(c-\frac{e}{2}+\frac{a}{2}+\frac{3}{2})}+\{(a-1)(a+2)+c(a-c+3)-e(2c-e+3)\}\frac{\Gamma(c+2-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2}+\frac{1}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}+\frac{1}{2})\Gamma(c-\frac{e}{2}+\frac{a}{2}+2)}\right.}{\int_0^1 x^{c-1}\,(1-x)^{c-e+i}[.1+\alpha x+\beta(1-x)]^{-2c+e-41}dx}$$

=0, elsewhere

$=1, \int_0^1 f(x)dx = 1$

Where f(x) = $F_1^2(a,-a,e{:}\frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)})$

Third Formula

$$\int_0^1 x^{c-1}\,(1-x)^{c-e+1}[.1+\alpha x+\beta(1-x)]^{-2c+e-2}F_1^2(a,1-a,e{:}\frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)})dx = \frac{\Gamma(e)\Gamma(c-e+2)\Gamma(e-c-1))\Gamma(c)}{2^{2a+2}(1+\alpha)^{c}(1+\beta)^{c-e+2}\Gamma(e-a)\Gamma(e-c)\Gamma(2c-e-a+2)}x[\{(a-1)(a+2)-2c(c+2)-e(3c-e+3)\}\frac{\Gamma(c-\frac{e}{2}-\frac{a}{2}+1)\Gamma(\frac{e}{2}-\frac{a}{2}+\frac{1}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}+\frac{1}{2})-\Gamma(c-\frac{e}{2}+\frac{a}{2}+2)}+\{-a(a+1)-e(c-e+1)\}\frac{\Gamma(c+\frac{3}{2}-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}+1)\Gamma(c-\frac{e}{2}+\frac{a}{2}+\frac{3}{2})}] \quad \text{..........................(4.2.3)}$$

Provided Re(c)>0,Re(c-e+2)>0, Re(e)>0. Also the constants α and β are such that none of the expressions 1+α,1+β and 1+αx+β(1-x), 0≤ x ≤1, is not zero.
So by well known definition of probability distributions we have:

$$F(x)=\frac{\frac{\Gamma(e)\Gamma(c-e+2)\Gamma(e-c-1))\Gamma(c)}{2^{2a+2}(1+\alpha)^c(1+\beta)^{c-e+2}\Gamma(e-a)\Gamma(e-c)\Gamma(2c-e-a+2)} x \left[\{(a-1)(a+2)-2c(c+2)-e(3c-e+3)\}\frac{\Gamma\left(c-\frac{e}{2}-\frac{a}{2}+1\right)\Gamma\left(\frac{e}{2}-\frac{a}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{e}{2}+\frac{a}{2}+\frac{1}{2}\right)-\Gamma\left(c-\frac{e}{2}+\frac{a}{2}+2\right)}+\{-a(a+1)-e(c-e+1)\}\frac{\Gamma\left(c+\frac{3}{2}-\frac{e}{2}-\frac{a}{2}\right)\Gamma\left(\frac{e}{2}-\frac{a}{2}\right)}{\Gamma\left(\frac{e}{2}+\frac{a}{2}+1\right)\Gamma\left(c-\frac{e}{2}+\frac{a}{2}+\frac{3}{2}\right)}\right]}{\int_0^1 x^{c-1}(1-x)^{c-e+i}[.1+\alpha x+\beta(1-x)]^{-2c+e-4}dx}$$

=0, elsewhere

=1, $\int_0^1 f(x)dx = 1$

Where f(x) = $F_1^2\left(a, 1-a;\ e: \frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)}\right)$

Fourth Formula

$$\int_0^1 x^{c-1}(1-x)^{c-e}[.1+\alpha x+\beta(1-x)]^{-2c+e-1}F_1^2\left(a,-2-a,e:\frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)}\right)dx = \frac{\Gamma(e)\Gamma(c-e+1)\Gamma(c)}{2^{2a+3}(1+\alpha)^c(1+\beta)^{c-e+1}\Gamma(e-a)\Gamma(2c-e-a+1)} x \left[\{e(2c-e+1)+a(a-c+1)\}\frac{\Gamma(c+\frac{1}{2}-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}+1)-\Gamma(c-\frac{e}{2}+\frac{1}{2a}+\frac{3}{2})}+\{e(2c-e+1)+(a+2)(a+c+1)\}\frac{\Gamma(c+1-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2}+\frac{1}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}+\frac{3}{2})\Gamma(c-\frac{e}{2}+\frac{a}{2}+2)}\right] \quad \text{.........................(4.2.4)}$$

Provided Re(c)>0,Re(c-e+1)>0, Re(e)>0. Also the constants α and β are such that none of the expressions 1+α,1+β and 1+αx+β(1-x), 0≤ x ≤1, is not zero.
So by well known definition of probability distributions we have:

$$F(x)=\frac{\left[\frac{\Gamma(e)\Gamma(c-e+1)\Gamma(c)}{2^{2a+3}(1+\alpha)^c(1+\beta)^{c-e+1}\Gamma(e-a)\Gamma(2c-e-a+1)} x\ \{e(2c-e+1)+a(a-c+1)\}\frac{\Gamma\left(c+\frac{1}{2}-\frac{e}{2}-\frac{a}{2}\right)\Gamma\left(\frac{e}{2}-\frac{a}{2}\right)}{\Gamma\left(\frac{e}{2}+\frac{a}{2}+1\right)-\Gamma\left(c-\frac{e}{2}+\frac{1}{2a}+\frac{3}{2}\right)}+\{e(2c-e+1)+(a+2)(a+c+1)\}\frac{\Gamma(c+1-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2}+\frac{1}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}+\frac{3}{2})\Gamma(c-\frac{e}{2}+\frac{a}{2}+2)}\right]}{\int_0^1 x^{c-1}(1-x)^{c-e+i}[.1+\alpha x+\beta(1-x)]^{-2c+e-4}dx}$$

=0, elsewhere

=1, $\int_0^1 f(x)dx = 1$

Where f(x) = $F_1^2\left(a,-2-a;\ e: \frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)}\right)$

Fifth Formula

$$\int_0^1 x^{c-1}(1-x)^{c-e+3}[.1+\alpha x+\beta(1-x)]^{-2c+e-4}F_1^2\left(a,2-a,e:\frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)}\right)dx = \frac{\Gamma(e)\Gamma(c-e+4)\Gamma(e-c-3)\Gamma(c)\Gamma(a-1)}{2^{2a+1}(1+\alpha)^c(1+\beta)^{c-e+4}\Gamma(e-a)\Gamma(e-c)\Gamma(a)\Gamma(2c-e-a+4)} x\ [\{-a(a-1)(a+e-$$

3)+c(a+c)} $\frac{\Gamma(c+2-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2}+\frac{1}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}-\frac{1}{2})-\Gamma(c-\frac{e}{2}+\frac{a}{2}+1)}$ **+{(a-1)(a-e+1)+c(a-c-**

2)} $\frac{\Gamma(c-\frac{e}{2}-\frac{a}{2}+\frac{5}{2})\Gamma(\frac{e}{2}-\frac{a}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}-1)\Gamma(c-\frac{e}{2}+\frac{a}{2}+\frac{3}{2})}$ **]..........................(4.2.5)**

Provided Re(c)>0,Re(c-e+4)>0, Re(e)>0. Also the constants α and β are such that none of the expressions 1+α,1+β and 1+αx+β(1-x), 0≤ x ≤1, is not zero.
So by well known definition of probability distributions we have:

$$F(x)=\frac{\left[\frac{\Gamma(e)\Gamma(c-e+4)\Gamma(e-c-3)\Gamma(c)\Gamma(a-1)}{2^{2a+1}(1+\alpha)^{c}(1+\beta)^{c-e+4}\Gamma(e-a)\Gamma(e-c)\Gamma(a)\Gamma(2c-e-a+4)}\text{x}\ \{-a(a-1)(a+e-3)+c(a+c)\}\frac{\Gamma\left(c+2-\frac{e}{2}-\frac{a}{2}\right)\Gamma\left(\frac{e}{2}-\frac{a}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{e}{2}+\frac{a}{2}-\frac{1}{2}\right)-\Gamma\left(c-\frac{e}{2}+\frac{a}{2}+1\right)}+\{(a-1)(a-e+1)+c(a-c-2)\}\frac{\Gamma(c-\frac{e}{2}-\frac{a}{2}+\frac{5}{2})\Gamma(\frac{e}{2}-\frac{a}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}-1)\Gamma(c-\frac{e}{2}+\frac{a}{2}+\frac{3}{2})}\right]}{\int_0^1 x^{c-1}\,(1-x)^{c-e+i}[.1+\alpha x+\beta(1-x)]^{-2c+e-4}dx}$$

=0, elsewhere

=1, $\int_0^1 f(x)dx=1$

Where f(x) = $F_1^2(a,-a;\ e:\frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)})$

Sixth Formula

$\int_0^1 x^{c-1}\,(1-x)^{c-e+3}[.1+\alpha x+\beta(1-x)]^{-2c+e-4}F_1^2(a,3-a,e:\frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)})dx\ =$
$\frac{\Gamma(e)\Gamma(c-e+4)\Gamma(e-c-3)\Gamma(c)\Gamma(a-2)}{2^{2a+2}(1+\alpha)^{c}(1+\beta)^{c-e+4}\Gamma(e-a)\Gamma(e-c)\Gamma(a)\Gamma(2c-e-a+4)}$ **x [{-(a-1)(a-2)+c(2c-**

e+2)} $\frac{\Gamma(c+2-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2}+\frac{1}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}-\frac{3}{2})-\Gamma(c-\frac{e}{2}+\frac{a}{2}+1)}$ **+{(a-1)(a-2)-c(e-**

2)} $\frac{\Gamma(c-\frac{e}{2}-\frac{a}{2}+\frac{5}{2})\Gamma(\frac{e}{2}-\frac{a}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}-1)\Gamma(c-\frac{e}{2}+\frac{a}{2}+\frac{1}{2})}$ **]..........................(4.2.6)**

Provided Re(c)>0,Re(c-e+4)>0, Re(e)>0. Also the constants α and β are such that none of the expressions 1+α,1+β and 1+αx+β(1-x), 0≤ x ≤1, is not zero.
So by well known definition of probability distributions we have:

$$F(x)=\frac{\left[\frac{\Gamma(e)\Gamma(c-e+4)\Gamma(e-c-3)\Gamma(c)\Gamma(a-2)}{2^{2a+2}(1+\alpha)^{c}(1+\beta)^{c-e+4}\Gamma(e-a)\Gamma(e-c)\Gamma(a)\Gamma(2c-e-a+4)}\text{x}\ \{-(a-1)(a-2)+c(2c-e+2)\}\frac{\Gamma(c+2-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2}+\frac{1}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}-\frac{3}{2})-\Gamma(c-\frac{e}{2}+\frac{a}{2}+1)}+\{(a-1)(a-2)-c(e-2)\}\frac{\Gamma(c-\frac{e}{2}-\frac{a}{2}+\frac{5}{2})\Gamma(\frac{e}{2}-\frac{a}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}-1)\Gamma(c-\frac{e}{2}+\frac{a}{2}+\frac{1}{2})}\right]}{\int_0^1 x^{c-1}\,(1-x)^{c-e+i}[.1+\alpha x+\beta(1-x)]^{-2c+e-4}dx}$$

=0, elsewhere

=1, $\int_0^1 f(x)dx=1$

Where f(x) = $F_1^2(a, 3-a;\ e: \frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)})$

Seventh Formula

$$\int_0^1 x^{c-1}(1-x)^{c-e+3}[.1+\alpha x+\beta(1-x)]^{-2c+e-4}F_1^2(a,4-a,e:\frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)})dx = \frac{\Gamma(e)\Gamma(c-e+4)\Gamma(e-c-3)\Gamma(c)\Gamma(a-3)}{2^{2a-3}(1+\alpha)^c(1+\beta)^{c-e+4}\Gamma(e-a)\Gamma(e-c)\Gamma(a)\Gamma(2c-e-a+4)} x[\{e(2c-e)-(a-6)(a-c+e)-c-11\}\frac{\Gamma(c+2-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2}+\frac{1}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}-\frac{3}{2})\Gamma(c-\frac{e}{2}+\frac{a}{2})}+\{(-e(2c-e+a+2)+(a+3)(a+c+1)-6a\}\frac{\Gamma(c-\frac{e}{2}-\frac{a}{2}+\frac{5}{2})\Gamma(\frac{e}{2}-\frac{a}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}-2)\Gamma(c-\frac{e}{2}+\frac{a}{2}+\frac{1}{2})}]\ldots\ldots\ldots\ldots(4.2.7)$$

Provided Re(c)>0,Re(c-e+4)>0, Re(e)>0. Also the constants α and β are such that none of the expressions 1+α,1+β and 1+αx+β(1-x), 0≤ x ≤1, is not zero.
So by well known definition of probability distributions we have:

$$F(x)=\frac{[\frac{\Gamma(e)\Gamma(c-e+4)\Gamma(e-c-3)\Gamma(c)\Gamma(a-3)}{2^{2a-3}(1+\alpha)^c(1+\beta)^{c-e+4}\Gamma(e-a)\Gamma(e-c)\Gamma(a)\Gamma(2c-e-a+4)} x \{e(2c-e)-(a-6)(a-c+e)-c-11\}\frac{\Gamma(c+2-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2}+\frac{1}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}-\frac{3}{2})\Gamma(c-\frac{e}{2}+\frac{a}{2})}+\{(-e(2c-e+a+2)+(a+3)(a+c+1)-6a\}\frac{\Gamma(c-\frac{e}{2}-\frac{a}{2}+\frac{5}{2})\Gamma(\frac{e}{2}-\frac{a}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}-2)\Gamma(c-\frac{e}{2}+\frac{a}{2}+\frac{1}{2})}]}{\int_0^1 x^{c-1}(1-x)^{c-e+i}[.1+\alpha x+\beta(1-x)]^{-2c+e-4}dx}$$

=0, elsewhere

=1, $\int_0^1 f(x)dx = 1$

Where f(x) = $F_1^2(a, 4-a;\ e: \frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)})$

Eighth Formula

$$\int_0^1 x^{c-1}(1-x)^{c-e}[.1+\alpha x+\beta(1-x)]^{-2c+e-1}F_1^2(a,4-a,e:\frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)})dx = \frac{\Gamma(e)\Gamma(c-e+1)\Gamma(c-3)\Gamma(c)\Gamma(a-3)}{2^{2a-3}(1+\alpha)^c(1+\beta)^{c-e+4}\Gamma(e-a)\Gamma(a)\Gamma(2c-e-a+1)} x[\{(a+3)(1+a-c)+e(2c-e+1)-6a\}\frac{\Gamma(c+\frac{1}{2}-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}-2)\Gamma(c-\frac{e}{2}+\frac{a}{2}-\frac{3}{2})}+\{-(a-7)(a+c-2)-e(2c-e+1)-3(a-1)\}\frac{\Gamma(c+1-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2}+\frac{1}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}-\frac{3}{2})\Gamma(c-\frac{e}{2}+\frac{a}{2}-1)}]\ldots\ldots\ldots\ldots(4.2.8)$$

Provided Re(c-3)>0,Re(c-e+1)>0, Re(e)>0. Also the constants α and β are such that none of the expressions 1+α,1+β and 1+αx+β(1-x), 0≤ x ≤1, is not zero.
So by well known definition of probability distributions we have:

$$F(x)=\frac{[\frac{\Gamma(e)\Gamma(c-e+1)\Gamma(c-3)\Gamma(c)\Gamma(a-3)}{2^{2a-3}(1+\alpha)^c(1+\beta)^{c-e+1}\Gamma(e-a)\Gamma(a)\Gamma(2c-e-a+1)} x [\{(a+3)(1+a-c)+e(2c-e+1)-6a\}\frac{\Gamma(c+\frac{1}{2}-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}-2)\Gamma(c-\frac{e}{2}+\frac{a}{2}-\frac{3}{2})}+\{-(a-7)(a+c-2)-e(2c-e+1)-3(a-1)\}\frac{\Gamma(c+1-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2}+\frac{1}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}-\frac{3}{2})\Gamma(c-\frac{e}{2}+\frac{a}{2}-1)}]}{\int_0^1 x^{c-1}(1-x)^{c-e}[.1+\alpha x+\beta(1-x)]^{-2c+e-4}dx}$$

=0, elsewhere

=1, $\int_0^1 f(x)dx = 1$

Where f(x) = $F_1^2(a, 4-a;\ e: \frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)})$

Ninth Formula

$$\int_0^1 x^{c-1}(1-x)^{c-e-1}[1+\alpha x+\beta(1-x)]^{-2c+e} F_1^2(a, 3-a;\ e: \frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)})dx = \frac{\Gamma(e)\Gamma(c-e)\Gamma(c-3)\Gamma(a-2)}{2^{2a-2}(1+\alpha)^c(1+\beta)^{c-e}\Gamma(e-a)\Gamma(a)\Gamma(2c-e-a)} \text{x}[\{(a-1)(a-2)+(e-c)(2c-e-2)\}\frac{\Gamma(c-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2}+\frac{1}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}-\frac{3}{2})\Gamma(c-\frac{e}{2}+\frac{a}{2}-1)}+\{(a-1)(a-2)+(e-2)(c-e)\}\frac{\Gamma(c+\frac{1}{2}-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}-1)\Gamma(c-\frac{e}{2}+\frac{a}{2}-\frac{3}{2})}] \quad \text{.........................(4.2.9)}$$

Provided Re(c-3)>0,Re(c-e)>0, Re(e)>0. Also the constants α and β are such that none of the expressions 1+α,1+β and 1+αx+β(1-x), 0≤ x ≤1, is not zero.
So by well known definition of probability distributions we have:

$$F(x)=\frac{[\frac{\Gamma(e)\Gamma(c-e)\Gamma(c-3)\Gamma(a-2)}{2^{2a-2}(1+\alpha)^c(1+\beta)^{c-e}\Gamma(e-a)\Gamma(a)\Gamma(2c-e-a)}\text{x}\ \{(a-1)(a-2)+(e-c)(2c-e-2)\}\frac{\Gamma(c-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2}+\frac{1}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}-\frac{3}{2})\Gamma(c-\frac{e}{2}+\frac{a}{2}-1)}+\{(a-1)(a-2)+(e-2)(c-e)\}\frac{\Gamma(c+\frac{1}{2}-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}-1)\Gamma(c-\frac{e}{2}+\frac{a}{2}-\frac{3}{2})}]}{\int_0^1 x^{c-1}(1-x)^{c-e-1}[1+\alpha x+\beta(1-x)]^{-2c+e}dx}$$

=0, elsewhere

=1, $\int_0^1 f(x)dx = 1$

Where f(x) = $F_1^2(a, 3-a;\ e: \frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)})$

Tenth Formula

$$\int_0^1 x^{c-1}(1-x)^{c-e-2}[1+\alpha x+\beta(1-x)]^{-2c+e+1} F_1^2(a, 2-a;\ e: \frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)})dx = \frac{\Gamma(e)\Gamma(c-e-1)\Gamma(c-3)\Gamma(a-1)}{2^{2a-1}(1+\alpha)^c(1+\beta)^{c-e-1}\Gamma(e-a)\Gamma(a)\Gamma(2c-e-a-1)} \text{x}[\{-(a-1)(a+1)+c(a+c-2)-e(2c-e-1)\}\frac{\Gamma(c-\frac{1}{2}-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}-1)\Gamma(c-\frac{1}{2}e+\frac{a}{2}-\frac{3}{2})}+\{(a-1)(a-3)-c(c-a)+e(2c-e-1)\}\frac{\Gamma(c-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2}+\frac{1}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}-\frac{1}{2})\Gamma(c-\frac{e}{2}+\frac{a}{2}-1)}] \quad \text{.........................(4.2.10)}$$

Provided Re(c-3)>0,Re(c-e-1)>0, Re(e)>0. Also the constants α and β are such that none of the expressions 1+α,1+β and 1+αx+β(1-x), 0≤ x ≤1, is not zero.
So by well known definition of probability distributions we have:

$$F(x)=\frac{[\frac{\Gamma(e)\Gamma(c-e-1)\Gamma(c-3)\Gamma(a-1)}{2^{2a-1}(1+\alpha)^c(1+\beta)^{c-e-1}\Gamma(e-a)\Gamma(a)\Gamma(2c-e-a-1)}\text{x}\ [\{-(a-1)(a+1)+c(a+c-2)-e(2c-e-1)\}\frac{\Gamma(c-\frac{1}{2}-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}-1)\Gamma(c-\frac{1}{2}e+\frac{a}{2}-\frac{3}{2})}+\{(a-1)(a-3)-c(c-a)+e(2c-e-1)\}\frac{\Gamma(c-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2}+\frac{1}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}-\frac{1}{2})\Gamma(c-\frac{e}{2}+\frac{a}{2}-1)}]}{\int_0^1 x^{c-1}(1-x)^{c-e-2}[1+\alpha x+\beta(1-x)]^{-2c+e+1}dx}$$

=0, elsewhere

$=1, \int_0^1 f(x)dx = 1$

Where f(x) = $F_1^2(a, 2-a;\ e: \frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)})$

Eleventh Formula

$$\int_0^1 x^{c-1}(1-x)^{c-e-3}[1+\alpha x+\beta(1-x)]^{-2c+e+2} F_1^2(a, 1-a;\ e: \frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)})dx = \frac{\Gamma(e)\Gamma(c-e-2)\Gamma(c-3)}{2^{2a}(1+\alpha)^c(1+\beta)^{c-e-2}\Gamma(e-a)\Gamma(2c-e-a-2)} x[\{-(a+2)(a-3)+3c(c-3)-e(3c-e-4)\}\frac{\Gamma(c-1-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2}+\frac{1}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}-\frac{1}{2})\Gamma(c-\frac{1}{2}e+\frac{a}{2}-1)}+\{-(a+1)(a-2)+c(c-3)+e(e-c)\}\frac{\Gamma(c-\frac{1}{2}-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2})\Gamma(c-\frac{e}{2}+\frac{a}{2}-\frac{3}{2})}] \quad ....(4.2.11)$$

Provided Re(c-3)>0,Re(c-e-2)>0, Re(e)>0. Also the constants α and β are such that none of the expressions 1+α,1+β and 1+αx+β(1-x), 0≤ x ≤1, is not zero.
So by well known definition of probability distributions we have:

$$F(x)=\frac{[\frac{\Gamma(e)\Gamma(c-e-2)\Gamma(c-3)}{2^{2a}(1+\alpha)^c(1+\beta)^{c-e-2}\Gamma(e-a))\Gamma(2c-e-a-2)} x [\{-(a+2)(a-3)+3c(c-3)-e(3c-e-4)\}\frac{\Gamma(c-1-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2}+\frac{1}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}-\frac{1}{2})\Gamma(c-\frac{1}{2}e+\frac{a}{2}-1)}+\{-(a+1)(a-2)+c(c-3)+e(e-c)\}\frac{\Gamma(c-\frac{1}{2}-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2})\Gamma(c-\frac{e}{2}+\frac{a}{2}-\frac{3}{2})}]}{\int_0^1 x^{c-1}(1-x)^{c-e-3}[1+\alpha x+\beta(1-x)]^{-2c+e+2}dx}$$

=0, elsewhere

$=1, \int_0^1 f(x)dx = 1$

Where f(x) = $F_1^2(a, 1-a;\ e: \frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)})$

Twelfth Formula

$$\int_0^1 x^{c-1}(1-x)^{c-e-3}[1+\alpha x+\beta(1-x)]^{-2c+e+2} F_1^2(a, -a;\ e: \frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)})dx = \frac{\Gamma(e)\Gamma(c-e-2)\Gamma(c-2)}{2^{2a+1}(1+\alpha)^c(1+\beta)^{c-e-2}\Gamma(e-a)\Gamma(2c-e-a-2)} x[\{-(a+1)(a+2)+a(c-e)+c(c-3)\}\frac{\Gamma(c-1-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2}+\frac{1}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}+\frac{1}{2})\Gamma(c-\frac{e}{2}+\frac{a}{2}-1)}+\{-(a+1)(a-2)-a(c-e)+c(c-3)\}\frac{\Gamma(c-\frac{1}{2}-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2})\Gamma(c-\frac{e}{2}+\frac{a}{2}-\frac{1}{2})}] \quad ....(4.2.12)$$

Provided Re(c)>2,Re(c-e-2)>0, Re(e)>0. Also the constants α and β are such that none of the expressions 1+α,1+β and 1+αx+β(1-x), 0≤ x ≤1, is not zero.
So by well known definition of probability distributions we have:

$$F(x)=\frac{[\frac{\Gamma(e)\Gamma(c-e-2)\Gamma(c-2)}{2^{2a+1}(1+\alpha)^c(1+\beta)^{c-e-2}\Gamma(e-a))\Gamma(2c-e-a-2)} x [\{-(a+1)(a+2)+a(c-e)+c(c-3)\}\frac{\Gamma\left(c-1-\frac{e}{2}-\frac{a}{2}\right)\Gamma\left(\frac{e}{2}-\frac{a}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{e}{2}+\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{e}{2}+\frac{a}{2}-1\right)}+\{-(a+1)(a-2)-a(c-e)+c(c-3)\}\frac{\Gamma(c-\frac{1}{2}-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2})\Gamma(c-\frac{e}{2}+\frac{a}{2}-\frac{1}{2})}]}{\int_0^1 x^{c-1}(1-x)^{c-e-3}[1+\alpha x+\beta(1-x)]^{-2c+e+2}dx}$$

=0, elsewhere

=1, $\int_0^1 f(x)dx = 1$

Where f(x) = $F_1^2(a, -a;\ e: \frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)})$

Thirteen Formula

$$\int_0^1 x^{c-1}(1-x)^{c-e-3}[1+\alpha x+\beta(1-x)]^{-2c+e+2} F_1^2(a, -1;\ e: \frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)})dx = \frac{\Gamma(e)\Gamma(c-e-2)\Gamma(c-1)}{2^{2a+2}(1+\alpha)^c(1+\beta)^{c-e-2}\Gamma(e-a)\Gamma(2c-e-a-2)} \mathrm{x}[\{-(a-1)(a+2)+(c-1)(2c-e)-2c\}\frac{\Gamma(c-1-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2}+\frac{1}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}+\frac{1}{2})\Gamma(c-\frac{e}{2}+\frac{a}{2})} + \{-a(a+1)+e(c-1)\}\frac{\Gamma(c-\frac{1}{2}-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}+1)\Gamma(c-\frac{e}{2}+\frac{a}{2}-\frac{1}{2})}] \ldots\ldots\ldots\ldots(4.2.13)$$

Provided Re(c)>1,Re(c-e-2)>0, Re(e)>0. Also the constants α and β are such that none of the expressions 1+α,1+β and 1+αx+β(1-x), 0≤ x ≤1, is not zero.
So by well known definition of probability distributions we have:

$$F(x) = \frac{[\frac{\Gamma(e)\Gamma(c-e-2)\Gamma(c-1)}{2^{2a+1}(1+\alpha)^c(1+\beta)^{c-e-2}\Gamma(e-a))\Gamma(2c-e-a-2)} \mathrm{x} [\{-(a-1)(a+2)+(c-1)(2c-e)-2c\}\frac{\Gamma(c-1-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2}+\frac{1}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}+\frac{1}{2})\Gamma(c-\frac{e}{2}+\frac{a}{2})} + \{-a(a+1)+e(c-1)\}\frac{\Gamma(c-\frac{1}{2}-\frac{e}{2}-\frac{a}{2})\Gamma(\frac{e}{2}-\frac{a}{2})}{\Gamma(\frac{e}{2}+\frac{a}{2}+1)\Gamma(c-\frac{e}{2}+\frac{a}{2}-\frac{1}{2})}]}{\int_0^1 x^{c-1}(1-x)^{c-e-3}[1+\alpha x+\beta(1-x)]^{-2c+e+2}dx}$$

=0, elsewhere

=1, $\int_0^1 f(x)dx = 1$

Where f(x) = $F_1^2(a, -1-a;\ e: \frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)})$

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