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Study on New Contractive Integral Type Conditions and Fixed Point Theorems in One Metrics

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Abstract:

Fixed point theory is one of the most fruitful and applicable topics of nonlinear analysis, which is widely used not only in other mathematical theories, but also in many practical problems of natural sciences and engineering. The Banach contraction mapping principle is indeed the most popular result of metric fixed point theory. This principle has many application in several domains, such as differential equations, functional equations, integral equations, economics, wild life, and several others. The aim of this paper is to extend the concept of F. Khojasteh, Z. Goodarzi and A. Razani to some new contractive conditions of integral type in cone metric space.

Key words: Cone Metric Space, Contractive Conditions, Fixed Point.

1. Introduction: The concept of cone metric space was introduced by Huang and Zhang [1] in 2007 and some fixed point theorems was proved. Initially Branciari [2] introduced the contractive condition of integral type and extended Banach fixed point theorem. Later on F. Khojasteh, Z. Goodarzi and A. Razani [3] gave the concept of cone integrable function and proved Branciari's theorem in cone metric space. The aim of this paper is to extend the concept of [3], to some new contractive conditions of integral type in cone metric space.

The following definitions and lemmas are useful for us to prove the main results.

Definition 1.1[1]**:** Let E be a real Banach space and P a subset of E. P is called a cone if the following hold.

(1) P is closed, non-empty and $P \neq \{0\}$.

(2) If $a, b \in R$ and $a, b \geq 0$, then $ax + by \in P$, $\forall\ x, y \in P$.

(3) $x \in P$ and $-x \in P$ implies $x = 0$.

Let $P \subseteq \mathsf{E}$ be a cone. We define a partial ordering with respect to P as $x \leq y$ if and only if $y - x \in P$ and $x < y$ will imply that $x \leq y$ but $x \neq y$, while $x << y$ will mean that $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P.

The cone P is called normal if there is a number $M > 0$ such that $0 \leq x < y$ implies $\|x\| \leq M\|y\|$ $\forall\ x, y \in \mathsf{E}$. The least positive number M is called the normal constant.

Example: Suppose $\mathsf{E} = R^2$, $P = \{(x, y) \in \mathsf{E} \mid x, y \geq 0\}$, $X = R$. Let $d : X \times X \to \mathsf{E}$ be defined as $d(x, y) = (b|x - y|, |x - y|)$ where $b \in R$ and $b \geq 0$. Then (X, d) is cone metric space.

Definition 1.2[1]**:** Let (X, d) be a cone metric space and let $\{x_n\}$ be a sequence in X. Then

(1) $\{x_n\}$ is said to converges to some $x \in X$ if for every $c \in \mathsf{E}$ with $0 << c$, $\exists$ a natural number N such that $\forall\ n \geq N$, $d(x_n, x) << c$.

(2) $\{x_n\}$ is said to be Cauchy sequence if for every $c \in \mathsf{E}$ with $0 << c$, $\exists$ a natural number N such that $\forall\ m, n \geq N$, $d(x_n, x_m) << c$. $\forall x, y \in P$

(3) A cone metric space (X, d) is complete if every Cauchy sequence is convergent.

Definition 1.3[3]**:** Let P be a normal cone in E and $\alpha, \beta \in \mathsf{E}$ where $\alpha < \beta$. Then we define

$$[\alpha, \beta] = \{x \in \mathsf{E} : s\beta + (1 - s)\alpha,\ s \in [0, 1]\},$$

$[\alpha, \beta) = \{x \in E : s\beta + (1-s)\alpha,\ s \in [0, 1)\}$.

Definition 1.4[3]: The set $P_1 = \{\alpha = x_0, x_1, x_2, ..., x_n = \beta\}$ is called a partition of $[\alpha, \beta]$ if and only if the sets $\{[x_{j-1}, x_j)\}_{j=1}^{n}$ are pairwise disjoint and $[\alpha, \beta] = \{\cup_{j=1}^{n} [x_{j=1}, x_j)\} \cup \{\beta\}$.

Definition 1.5[3]: Let $P_1 = \{\alpha = x_0, x_1, x_2, ..., x_n = \beta\}$ be a partition of $[\alpha, \beta]$ and $\phi = [\alpha, \beta] \to P$ be an increasing function. We define cone lower sum and cone upper sum as

$$L_{n^-}^{con}(\phi, P_1) = \sum_{j=0}^{n-1} \phi(x_j) \|x_j - x_{j+1}\|,$$

$$U_n^{con}(\phi, P_1) = \sum_{j=0}^{n-1} \phi(x_{j+1}) \|x_j - x_{j+1}\|, \text{ respectively.}$$

The function ϕ is called cone integrable function on $[\alpha, \beta]$ if and only if for all partitions P_1 of $[\alpha, \beta]$

$$\lim_n L_n^{con}(\phi, P_1) = S^{con} = \lim_n U_n^{con}(\phi, P_1),$$

where S^{con} is unique. We shall write $S^{con} = \int_\alpha^\beta \phi\, dp$ or $\int_\alpha^\beta \phi(t)\, dp(t)$.

Lemma 1.1[3]: If $[\alpha, \beta] \subseteq [\alpha, \gamma]$ then $\int_\alpha^\beta \phi\, dp \le \int_\alpha^\gamma \phi\, dp$ for $\phi \in \ell^1(X, P)$

$\int_\alpha^\beta (a\phi_1 + b\phi_2) dp = a\int_\alpha^\beta \phi_1\, dp + b\int_\alpha^\beta \phi_2\, dp$ for $\phi_1, \phi_2 \in \ell^1(X, P)$ and $a, b \in R$

where $\ell^1(X, P)$ denotes the set all cone integrable functions.

Definition 1.6[3]: A function $\phi : P \to E$ is said to be subadditive cone integrable function if and only if $\forall\ \alpha, \beta \in P$

$$\int_0^{\alpha+\beta} \phi\, dp \le \int_0^\alpha \phi\, dp + \int_0^\beta \phi\, dp.$$

2. Main Results:

Theorem 2.1: Let (X, d) be a complete cone metric space with normal cone *P*. Let $\phi : P \to P$ be a nonvanishing and subadditive cone integrable map on each $[\alpha, \beta] \subset P$ for which $\int_0^\varepsilon \phi\, dp >> 0$, $\varepsilon >> 0$. Let $T : X \to X$ be a mapping such that

$$\int_0^{d(T(x),T(y))} \phi\, dp \le c\int_0^{d(x,T(y))+d(y,T(x))} \phi\, dp \text{ for each } x, y \in X,\ c \in \left(0, \frac{1}{2}\right).$$

Then *T* has a unique fixed point in *X*.

Proof: Let $x \in X$, choose $x_1 \in X$ such that $x_1 = T(x)$. Let $x_2 \in X$ be such that $x_2 = T(x)$. Continuing in this way we can define $x_n = T(x_{n-1}) = T^n(x)$ for $n = 1, 2, 3, ...$

$$\int_0^{d(x_{n+1}, x_n)} \phi\, dp = \int_0^{d(T(x_n), T(x_{n-1}))} \phi\, dp$$

$$\le c\int_0^{d(x_n, x_n) + d(x_{n-1}, x_{n+1})} \phi\, dp$$

$$\le c\int_0^{d(x_{n-1}, x_{n+1})} \phi\, dp$$

But $d(x_{n-1}, x_{n+1}) \le d(x_{n-1}, x_n) + d(x_n, x_{n+1})$, therefore

$$\int_0^{d(x_{n+1}, x_n)} \phi\, dp \le c\int_0^{d(x_{n-1}, x_n) + d(x_n, x_{n+1})} \phi\, dp$$

Since ϕ is cone subadditive, so

$$\int_0^{d(x_{n+1},x_n)} \phi\, dp \le c\int_0^{d(x_{n-1},x_n)} \phi\, dp + c\int_0^{d(x_n,x_{n+1})} \phi\, dp$$

$$\Rightarrow \quad \int_0^{d(x_{n+1},x_n)} \phi\, dp \le \frac{c}{1-c}\int_0^{d(x_n,x_{n-1})} \phi\, dp = k\int_0^{d(x_n,x_{n-1})} \phi\, dp, \qquad \text{where } k=\frac{c}{1-c}$$

$$\vdots$$

$$\le k^n \int_0^{d(x_1,x_0)} \phi\, dp$$

$$\int_0^{d(x_{n+1},x_n)} \phi\, dp \le k^n \int_0^{d(T(x),x)} \phi\, dp$$

Since $0 \le k < 1$, and $\int_0^{\varepsilon} \phi\, dp >> 0$ for each $\varepsilon >> 0$, so

$$\lim_n \int_0^{d(x_{n+1},x_n)} \phi\, dp = 0,$$

which implies, that $\lim_n d(x_{n+1},x_n) = 0.$

To show $\{x_n\}$ is Cauchy sequence, we shall show that $\lim_{n\to\infty} d(T(x_{n+\rho}),T(x_n)) = 0$ for each positive integer ρ.

Let $\rho > 0$ be any integer. By triangular inequality

$$d(x_{n+\rho},x_n) \le d(x_{n+\rho},x_{n+\rho-1}) + d(x_{n+\rho-1},x_{n+\rho-2}) + \ldots + d(x_{n+1},x_n)$$

$$\int_0^{d(x_{n+\rho},x_n)} \phi\, dp \le \int_0^{d(x_{n+\rho},x_{n+\rho-1})+\ldots+d(x_{n+1},x_n)} \phi\, dp$$

$$\int_0^{d(T(x_{n+\rho+1}),T(x_n))} \phi\, dp = \int_0^{d(x_{n+\rho},x_n)} \phi\, dp \le \int_0^{d(x_{n+\rho},x_{n+\rho-1})+\ldots+d(x_{n+1},x_n)} \phi\, dp$$

Since ϕ is cone subadditive

$$\le \int_0^{d(x_{n+\rho},x_{n+\rho-1})} \phi\, dp + \int_0^{d(x_{n+\rho-1},x_{n+\rho-2})} \phi\, dp + \ldots + \int_0^{d(x_{n+1},x_1)} \phi\, dp$$

$$\le \left(k^{n+\rho-1} + k^{n+\rho-2} + \ldots + k^n\right)\int_0^{d(x_1,x_0)} \phi\, dp$$

$$\le \left(k^n + k^{n+1} + \ldots + k^{n+\rho-2} + k^{n+\rho-1}\right)\int_0^{d(T(x),x)} \phi\, dp$$

$$\le \frac{k^n}{1-k}\int_0^{d(T(x),x)} \phi\, dp$$

Letting $n \to \infty$, $\lim_{n\to\infty} \int_0^{d(T(x_{n+\rho+1}),T(x_n))} \phi\, dp = 0$.

Which implies that $\lim_{n\to\infty} d(T(x_{n+\rho}),T(x_n)) = 0$ for each positive integer ρ.

Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete cone metric space so $\{x_n\}$ is convergent to some $z \in X$. i.e. $\lim_n x_n = z$.

$$\int_0^{d(T(z),x_{n+1})} \phi\, dp = \int_0^{d(T(z),T(x_n))} \phi\, dp$$

$$\le c\int_0^{d(z,x_{n+1})+d(x_n,T(z))} \phi\, dp$$

$$\le c\int_0^{d(z,x_{n+1})} \phi\, dp + c\int_0^{d(x_n,T(z))} \phi\, dp$$

As $n \to \infty$

$$\int_0^{d(T(z),z)} \phi\, dp \le c \int_0^{d(z,T(z))} \phi\, dp$$

which implies that $d(T(z),z)=0$ i.e. $T(z)=z$.

Thus z is a fixed point of T.

Uniqueness: Let T has two fixed point z and w i.e. $T(z)=z$ and $T(w)=w$.

$$\int_0^{d(z,w)} \phi\, dp = \int_0^{d(T(z),T(w))} \phi\, dp \le c \int_0^{d(z,T(w))+d(w,T(z))} \phi\, dp$$

$$\le c \int_0^{d(z,w)} \phi\, dp + c \int_0^{d(w,z)} \phi\, dp$$

$$\Rightarrow \quad \int_0^{d(z,w)} \phi\, dp \le \frac{c}{1-c} \int_0^{d(z,w)} \phi\, dp = k \int_0^{d(z,w)} \phi\, dp \text{ where } k=\frac{c}{1-c}$$

Which implies that $d(z,w)=0$ i.e. $z=w$.

This shows that T has a unique fixed point in X.

Theorem 2.2: Let (X,d) be a complete cone metric space with normal cone P. Let $\phi: P \to P$ be a nonvanishing and subadditive cone integrable map on each $[\alpha,\beta] \subset P$ for which $\int_0^{\varepsilon} \phi\, dp >> 0$, $\varepsilon >> 0$. Let $T: X \to X$ be a mapping such that $\int_0^{d(T(x),T(y))} \phi\, dp \le a \int_0^{d(x,y)} \phi\, dp + b \int_0^{d(y,T(x))} \phi\, dp$. For $a,b \in R$ s.t. $a < 1-2b$ and $0 \le b < \frac{1}{2}$. Then T has unique fixed point.

Proof: Let $x \in X$, choose $x_1 \in X$ such that $x_1 = T(x)$. Let $x_2 \in X$ be such that $x_2 = T(x)$. Continuing in this way we can define $x_n = T(x_{n-1}) = T^n(x)$ for $n = 1, 2, 3, \ldots$

$$\int_0^{d(x_{n+1},x_n)} \phi\, dp = \int_0^{d(T(x_n),T(x_{n-1}))} \phi\, dp$$

$$\le a \int_0^{d(x_n,x_{n-1})} \phi\, dp + b \int_0^{d(x_{n-1},x_{n+1})} \phi\, dp$$

Using triangle inequality and cone subadditivity,

$$\le a \int_0^{d(x_n,x_{n-1})} \phi\, dp + b \int_0^{d(x_{n-1},x_n)} \phi\, dp + b \int_0^{d(x_n,x_{n+1})} \phi\, dp$$

$$\int_0^{d(x_{n+1},x_n)} \phi\, dp \le \frac{a+b}{1-b} \int_0^{d(x_n,x_{n-1})} \phi\, dp = k \int_0^{d(x_n,x_{n-1})} \phi\, dp, \quad \text{where } k=\frac{a+b}{1-b}$$

$$\vdots$$

$$\int_0^{d(x_{n+1},x_n)} \phi\, dp \le k^n \int_0^{d(x_1,x_0)} \phi\, dp = k^n \int_0^{d(T(x),x)} \phi\, dp$$

Since $k=\frac{a+b}{1-b}<1$ then as $n \to \infty$, $\lim_n \int_0^{d(x_{n+1},x_n)} \phi\, dp = 0$

Which implies that $\lim_n d(x_{n+1},x_n)=0$.

It is easy to show that $\{x_n\}$ is a Cauchy sequence (See previous theorem). Since X is complete cone metric space so there is some $z \in X$ such that $\lim_n x_n = z$.

Now, $$\int_0^{d(T(z),x_{n+1})} \phi\, dp = \int_0^{d(T(z),T(x_n))} \phi\, dp$$

$$\le a \int_0^{d(z,x_n)} \phi\, dp + b \int_0^{d(x_n,T(z))} \phi\, dp$$

As $n \to \infty$, $\int_0^{d(T(z),z)} \phi\, dp \le b \int_0^{d(z,T(z))} \phi\, dp$

Since $0 \le b < \frac{1}{2}$ then $\int_0^{d(T(z),z)} \phi\, dp = 0$ which implies that $d(T(z), z) = 0 \Rightarrow T(z) = z$.

Uniqueness: Let T has two fixed point z and w i.e. $T(z) = z$ and $T(w) = w$.

$$\int_0^{d(z,w)} \phi\, dp = \int_0^{d(T(z),T(w))} \phi\, dp$$

$$\le a \int_0^{d(z,w)} \phi\, dp + b \int_0^{d(w,T(z))} \phi\, dp$$

$$= (a+b) \int_0^{d(z,w)} \phi\, dp.$$

Since $0 < a+b < 1$ therefore

$$\int_0^{d(z,w)} \phi\, dp = 0$$

$$\Rightarrow \quad d(z,w) = 0$$

$$\Rightarrow \quad z = w.$$

It shows that T has a unique fixed point.

Theorem 2.3: Let (X, d) be a complete cone metric space with normal cone P. Let $\phi : P \to P$ be a nonvanishing and subadditive cone integrable map on each $[\alpha, \beta] \subset P$ for which $\int_0^{\varepsilon} \phi\, dp >> 0$, $\varepsilon >> 0$. Let $T : X \to X$ be a mapping such that

$\int_0^{d(T(x),T(y))} \phi\, dp \le c \int_0^{d(x,T(x))+d(y,T(y))} \phi\, dp$. For $c \in \left(0, \frac{1}{2}\right)$ then T has a unique fixed point in X.

Proof: Let $x \in X$, choose $x_1 \in X$ such that $x_1 = T(x)$. Let $x_2 \in X$ be such that $x_2 = T(x)$. Continuing in this way we can define $x_n = T(x_{n-1}) = T^n(x)$ for $n = 1, 2, 3, \ldots$

$$\int_0^{d(x_{n+1},x_n)} \phi\, dp = \int_0^{d(T(x_n),T(x_{n-1}))} \phi\, dp \le c \int_0^{d(x_n,x_{n+1})+d(x_{n-1},x_n)} \phi\, dp$$

$$\le c \int_0^{d(x_n,x_{n+1})} \phi\, dp + c \int_0^{d(x_n,x_{n-1})} \phi\, dp$$

$$\int_0^{d(x_{n+1},x_n)} \phi\, dp \le \frac{c}{1-c} \int_0^{d(x_n,x_{n-1})} \phi\, dp = k \int_0^{d(x_n,x_{n-1})} \phi\, dp$$

As in theorems (2.1), it is easy to prove that $\{x_n\}$ is a Cauchy sequence and completeness of X implies that there is some $z \in X$ such that $\lim_n x_n = z$.

Now, $\int_0^{d(T(z),x_{n+1})} \phi\, dp = \int_0^{d(T(z),T(x_n))} \phi\, dp$

$$\le c \int_0^{d(z,T(z))+d(x_n,x_{n+1})} \phi\, dp$$

$$\le c \int_0^{d(z,T(z))} \phi\, dp + c \int_0^{d(x_n,x_{n+1})} \phi\, dp$$

As $n \to \infty$, $\int_0^{d(T(z),z)} \phi\, dp \le c \int_0^{d(T(z),z)} \phi\, dp$ which implies that $d(T(z), z) \Rightarrow T(z) = z$.

Uniqueness: Let T has two fixed point z and w i.e. $T(z) = z$ and $T(w) = w$.

$$\int_0^{d(z,w)} \phi\, dp = \int_0^{d(T(z),T(w))} \phi\, dp$$

$$\leq c\int_0^{d(z,T(z))+d(w,T(w))} \phi\, dp$$

$$\leq c\int_0^{d(z,T(z))} \phi\, dp + c\int_0^{d(w,T(w))} \phi\, dp = 0 \Rightarrow \quad d(z,w)=0 \;\Rightarrow\; z=w\,.$$

Theorem 2.4: Let (X, d) be a complete cone metric space with normal cone *P*. Let $\phi : P \to P$ be a nonvanishing and subadditive cone integrable map on each $[\alpha, \beta] \subset P$ for which $\int_0^{\varepsilon} \phi\, dp >> 0$, $\varepsilon >> 0$. Let $T : X \to X$ be a mapping such that

$$\int_0^{d(T(x),T(y))} \phi\, dp \leq c\int_0^{d(x,T(y))+d(y,T(x))+d(x,y)} \phi\, dp$$ For some $c \in \left(0, \frac{1}{3}\right)$ than T has a unique fixed point in *X*.

Proof: Let $x \in X$, define $x_{n+1} = T(x_n)$ for $n \geq 1$ and $x_1 = T(x_0) = T(x)$.

$$\int_0^{d(x_{n+1},x_n)} \phi\, dp = \int_0^{d(T(x_n),T(x_{n-1}))} \phi\, dp$$

$$\leq c\int_0^{d(x_n,x_n)+d(x_{n-1},x_{n+1})+d(x_n,x_{n-1})} \phi\, dp$$

$$\leq c\int_0^{d(x_{n-1},x_{n+1})} \phi\, dp + c\int_0^{d(x_n,x_{n-1})} \phi\, dp$$

Using triangular inequality and cone subadditivity.

$$\leq c\int_0^{d(x_{n-1},x_n)} \phi\, dp + c\int_0^{d(x_n,x_{n+1})} \phi\, dp + c\int_0^{d(x_n,x_{n-1})} \phi\, dp$$

$$\int_0^{d(x_{n+1},x_n)} \phi\, dp \leq \frac{2c}{1-c}\int_0^{d(x_n,x_{n-1})} \phi\, dp$$

$$\vdots$$

$$\leq \left(\frac{2c}{1-c}\right)^n \int_0^{d(x_1,x_0)} \phi\, dp = \left(\frac{2c}{1-c}\right)^n \int_0^{d(T(x),x)} \phi\, dp$$

If $0 < \frac{2c}{1-c} < 1$ i.e. $c < \frac{1}{3}$ then

$$\lim_n \int_0^{d(x_{n+1},x_n)} \phi\, dp = 0,$$

which implies that $\lim_n d(x_{n+1}, x_n) = 0$.

It is easy to prove that $\{x_n\}$ is Cauchy sequence. Since *X* is complete cone metric space so there is some $z \in X$ such that $\lim_n x_n = z$.

Now, $$\int_0^{d(T(z),x_{n+1})} \phi\, dp = \int_0^{d(T(z),T(x_n))} \phi\, dp$$

$$\leq c\int_0^{d(z,x_{n+1})+d(x_n,T(z))+d(z,x_n)} \phi\, dp$$

$$\leq c\int_0^{d(z,x_{n+1})} \phi\, dp + c\int_0^{d(x_n,T(z))} \phi\, dp + c\int_0^{d(z,x_n)} \phi\, dp$$

As $n \to \infty$, $\int_0^{d(T(z),z)} \phi\, dp \leq c\int_0^{d(z,T(z))} \phi\, dp$

Which implies that $d(T(z), z) = 0$. i.e. $T(z) = z$.

Hence *z* is a fixed point of *T*.

Uniqueness: Let *z* and *w* are two fixed points of *T*. i.e. $T(z) = z$ and $T(w) = w$.

$$\int_0^{d(z,w)} \phi\, dp = \int_0^{d(T(z),T(w))} \phi\, dp$$

$$\leq c\int_0^{d(z,T(w))+d(w,T(z))+d(z,w)} \phi\, dp$$

$$\int_0^{d(z,w)} \phi\, dp \leq c\int_0^{3d(z,w)} \phi\, dp\,.$$

Which is possible if $d(z,w)=0$ i.e. $z=w$.

Thus fixed point of T is unique.

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