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Mathematical Modeling and Analysis of Non-Newtonian Fluid Flow in Porous Media

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Abstract

Numerous technical, geophysical, medical, and industrial applications of non-Newtonian fluid flow in porous media have piqued the public's attention. These applications include groundwater hydrology, filtration, polymer processing, and better oil recovery. The nonlinear connection between shear stress and strain rate makes the flow behavior of non-Newtonian fluids across porous media more difficult than that of Newtonian fluids. A complex mathematical model is used in this research to investigate the non-Newtonian fluid's constant, incompressible, laminar flow through an isotropic, homogeneous porous media. The updated Darcy-Brinkman equations are derived by applying the appropriate constitutive relations to non-Newtonian fluids. It is possible to simplify the controlling PDEs to a system of nonlinear ODets by using similarity transformations. The answer is obtained by numerically solving the resultant equations, which are obtained by using traditional techniques to boundary value concerns. Flow features and velocity profiles are discussed in relation to critical physical properties like porosity, permeability, non-Newtonian fluid parameters, and Darcy number. In addition to paving the way for further theoretical and practical investigations, these findings clarify the nature of non-Newtonian fluid flow in porous materials.

Keywords: Non-Newtonian fluids, Porous media, Mathematical modeling, Similarity transformation, Numerical analysis

Introduction:

Fluid flow through porous media is a traditional topic in fluid mechanics with applications in many fields, including chemical, medicinal, environmental, and petroleum engineering, as well as material processing. The law of Darcy, which states that the relationship between fluid velocity and pressure gradient is linear, is often used in conventional flow models for porous media. Non-Newtonian fluids exhibit non-linear rheological behavior, hence this does not apply in their case, even if it does in cases of Newtonian fluids with low Reynolds numbers. Some fluids, such as paints, molten plastics, biological fluids, slurries, suspensions, or polymer solutions, do not adhere to Newton's law of viscosity. Their shear-thinning, shear-thickening, viscoplastic, or viscoelastic behavior is instead caused by the fact that their visible viscosity is rate-dependent. Additional modeling and analytical issues arise when these fluids move through porous media due to the complicated rheology of the fluid interacting with the microstructure of the medium.

Classical porous medium flow models have been extended by many academics in recent decades to include non-Newtonian phenomena. It has been proposed to include the inertial and viscous effects of the porous medium into an expansion of the Darcy equations, which includes the modified Darcy model, the Darcy-Forchheimer model, and the Brinkman-type. Investigational tools that are very desired include similarity analysis and numerical approaches; nevertheless, analytical solutions to the models may be hindered by the nonlinear nature of the governing equations.

A comprehensive mathematical model for the investigation of non-Newtonian fluid flow in porous media is the goal of this research work. By using the generalized constitutive relationship, which transforms the equations to similarity transformations, the complicated governing equations may be made more comprehensible. The quantitative results provide light on the ways in which the flow behavior is affected by non-Newtonian factors and the characteristics of porous media.

Mathematical Formulation

Using the fundamental conservation principles and pertinent rheological constitutive relations, a mathematical model is developed to investigate the flow phenomena of non-Newtonian fluids. Much of the influence on the momentum transfer mechanism comes from the interaction between the non-Newtonian fluid's complicated rheology and the resistance offered by the porous matrix. Consequently, the impacts of the porous media are included into the continuous way of approach. The physical model, governing equations, and constitutive equations relevant to the flow under investigation are shown here.

Physical Model and Assumptions

An incompressible two-dimensional steady-state laminar flow of a non-Newtonian fluid through an isotropic, homogenous, and saturated porous material is being considered in this research. Since the fluid entirely fills the pore space, it is further presumed that the porous matrix is stiff. The following assumptions are made to simplify the mathematical formulation while keeping the essential physics of the issue:

- The porous medium is homogeneous and isotropic with constant porosity and permeability.
- The flow regime is laminar, and inertial effects are moderate.
- Thermal effects, heat transfer, and chemical reactions are neglected.
- Body forces such as gravity are ignored, and only pressure-driven flow is considered.

These assumptions are commonly employed in porous media flow analysis and are appropriate for low to moderate Reynolds number flows encountered in many engineering and industrial applications.

Governing Equations

The flow may be described using the continuity and momentum equations, supposing the conditions specified before.

Continuity Equation

For an incompressible fluid, the conservation of mass is expressed as

$$\boldsymbol{\nabla} \cdot \boldsymbol{V} = 0 \qquad (1)$$

Where

$$\boldsymbol{V} = (\boldsymbol{u}, \boldsymbol{v}) \qquad (2)$$

is the velocity vector with u and v denoting the velocity components in the x- and y-directions, respectively.

Momentum Equation

The Brinkman-extended Darcy model is used to take porous medium resistance and viscous diffusion into consideration. For a non-Newtonian fluid passing through a porous material, the momentum equation is expressed as

$$\rho(V \cdot \nabla)V = -\nabla p + \nabla \cdot \tau - \frac{\mu eff}{K} V \qquad (3)$$

The Darcy resistance, which simulates the drag caused by the porous matrix on the fluid's velocity, is represented by the final term of the equation. This term is reduced to the traditional Navier-Stokes equations at permeability values that are quite big.

For two-dimensional flow, the momentum equations can be expressed component-wise as

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} - \frac{\mu_{eff}}{K} u \qquad (4)$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} - \frac{\mu_{eff}}{K} v \qquad (5)$$

Constitutive Relation for Non-Newtonian Fluid

A nonlinear connection between the shear stress and the rate of strain is a characteristic of the rheological behavior of non-Newtonian fluids. The constitutive equation may be expressed in its broadest form as

$$\boldsymbol{\tau} = f(\dot{\gamma}) \qquad (6)$$

where

$$\dot{\gamma} = \sqrt{2\mathbf{D}:\mathbf{D}} \qquad (7)$$

is the magnitude of the rate-of-strain tensor, and **D**is defined as

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$$\mathbf{D} = \frac{1}{2}(\nabla\mathbf{V} + (\nabla\mathbf{V})^T) \quad (8)$$

For many engineering applications, the power-law model provides a convenient and effective representation of non-Newtonian behavior. Accordingly, the shear stress is expressed as

$$\tau = K_1(\dot{\gamma})^n \quad (9)$$

where

- $K_1$is the consistency index,
- $n$is the flow behavior index.

The apparent viscosity $\mu_{app}$of the fluid is then given by

$$\mu_{app} = K_1(\dot{\gamma})^{n-1} \quad (10)$$

To control the non-Newtonian fluid flow through porous media, we may plug the constitutive relation into the momentum equations and get a closed system of nonlinear partial differential equations.

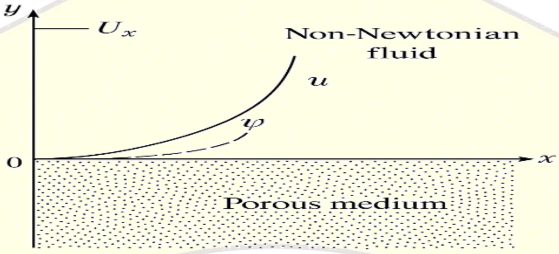

Figure 1: Statistical framework and spatial reference for the non-Newtonian transport of fluids through porous media

Similarity Transformation

Section 2 consist of nonlinear partial equations and are obtained by the joint action of non-Newtonian rheology and resistance of porous medium. Such equations are normally hard to analytically treat. This transformation allows conducting a systematic study of the flow properties and indicates the importance of several important dimensionless parameters.

Similarity Variables

To automatically satisfy the continuity equation for incompressible flow,

$$\nabla \cdot \mathbf{V} = 0, \quad (11)$$

a stream function $\psi(x, y)$is introduced such that

$$u = \frac{\partial\psi}{\partial y}, v = -\frac{\partial\psi}{\partial x} \quad (12)$$

where $u$and $v$denote the velocity components in the x- and y-directions, respectively. Substitution of these definitions into the continuity equation yields

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2\psi}{\partial x\,\partial y} - \frac{\partial^2\psi}{\partial y\,\partial x} = 0, \quad (13)$$

confirming that the continuity equation is identically satisfied.

Non-Dimensional Variables

To generalize the analysis and identify the key governing parameters, the following dimensionless variables are introduced:

$$\eta = y\left(\frac{U_0}{\nu_{eff}x}\right)^{1/2}, \psi = \left(\nu_{eff}U_0 x\right)^{1/2} f(\eta) \quad (14)$$

Using the above transformations, the velocity components become

$$u = \frac{\partial\psi}{\partial y} = U_0 f'(\eta) \quad (15)$$

$$v = -\frac{\partial\psi}{\partial x} = \frac{1}{2}\left(\frac{\nu_{eff}U_0}{x}\right)^{1/2} [\eta f'(\eta) - f(\eta)] \quad (16)$$

Transformation of Shear Rate and Stress

Substituting the similarity variables into the momentum equation for flow through a porous medium yields a nonlinear ordinary differential equation of the form

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$$(f')^n f'' + \beta f f'' - \lambda f' = 0 \qquad (17)$$

where

$$\beta = \frac{2n}{n+1} \qquad (18)$$

and

$$\lambda = \frac{\nu_{eff}}{K U_0} \qquad (19)$$

is the porous medium resistance parameter.

The non-Newtonian behavior is reflected through the power-law index n, while the effect of the porous medium is incorporated via the Darcy resistance parameter λ.

Dimensionless Parameters

The similarity transformation introduces several important dimensionless parameters that govern the flow behavior:

- **Darcy number**

$$Da = \frac{K}{L^2} \qquad (20)$$

Modified Reynolds number

$$Re = \frac{\rho U_0 L}{\mu_{eff}} \qquad (21)$$

which measures the ratio of inertial to viscous forces.

Boundary Conditions

The physical boundary conditions at the solid surface and far away from the surface are transformed into similarity form as

$$f(0) = 0, f'(0) = 0 \qquad (22)$$

$$f'(\infty) \to 1$$

where $f'(0) = 0$corresponds to the no-slip condition at the porous boundary, and $f'(\infty) = 1$represents the free-stream velocity condition.

Numerical Solution

A linked set of nonlinear ordinary differential equations is obtained from the flow model's governing equations; these equations are very nonlinear and include the interplay of several physical factors; thus, they cannot be solved analytically. Therefore, we use a numerical technique. In order to meet the specified boundary conditions, the transformed boundary value problem is solved using a Runge-Kutta approach coupled with a suitable iterative strategy. This kind of issue is well-suited to the Runge-Kutta technique because to its numerical stability, high precision, and effective handling of stiff and nonlinear systems. The stability of the approach is guaranteed across a large range of governing physical parameters, and the convergence of the numerical solution is closely monitored, verifying the dependability and robustness of the computational process.

The impact of critical physical characteristics, such as the porous medium's permeability parameter and the non-Newtonian flow behavior index, may be studied by computing velocity profiles. Fluid velocity and the thickness of the momentum boundary layer are both affected by the permeability parameter, which shows how the porous matrix acts as a barrier to fluid motion. A lower permeability value indicates a greater flow resistance, and vice versa. On the flip side, as the permeability of a material increases, the drag force exerted by the porous medium decreases, enabling the fluid to flow with more freedom and ultimately increasing the velocity across the boundary layer. The velocity distribution is also sensitive to changes in the non-Newtonian flow index. In shear-thinning fluids, the effective viscosity drops as the shear rate rises, allowing for higher flow velocities, whereas in shear-thickening fluids, the velocity field is reduced because of the greater resistance to deformation. Taken together, these findings provide light on the physical aspects of transport processes in flows across porous media by demonstrating how the velocity distribution is highly dependent on features of the medium and non-Newtonian fluid properties.

Results and Discussion

The mathematical results show that the flow behavior of porous media is significantly influenced by non-Newtonian fluid dynamics. When compared to Newtonian fluids, shear-thinning fluids will have a higher velocity, while shear-thickening fluids will have a higher flow resistance. While high porous resistance inhibits fluid flow, an increase in permeability improves the velocity field.

It is also revealed by the study that interrelation between non-Newtonian parameters and porous medium qualities are very important in the determination of flow patterns. The results are in agreement with theoretical and experimental findings published in the past.

Conclusion

This study presents a mathematical model and numerical simulation of non-Newtonian fluid flow across porous media, highlighting the relationship between fluid rheology and porous structural resistance. Then, by using similarity transformations and suitable numerical methods, the complex's governing equations might be reduced to a set of nonlinear ordinary differential equations. Variations in the properties of non-Newtonian fluids and the permeability of porous media impact the velocity profiles and the overall flow behavior, as is clearly seen by the calculated results. In order to manage momentum transfer in porous materials, the results show how important rheological effects are. Adding other physical processes like temperature effects, magnetic fields, and time-dependent flow conditions to the developed model might be a future area of investigation. It offers the theoretical groundwork for investigating transport systems that are not Newtonian.

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