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Similarity Analysis of Natural Convection Boundary-Layer Flow of Powell–Eyring Fluids Past a Vertical Plate

Mehul Tithlia, Asst. Prof. (Mathematics), Dolat –Usha Institute of Applied Sciences and Dhiru-Sarla Institute of Management & Commerce, Valsad, Gujarat

Abstract

The continuous laminar natural convection boundary-layer flow of an incompressible Powell-Eyring non-Newtonian fluid past a vertical flat plate is analysed mathematically. The boundary-layer and Boussinesq approximations are used to develop the governing nonlinear partial differential equations, which are made up of the continuity, momentum, and energy equations coupled by a nonlinear constitutive relation for the shear stress. The Powell–Eyring stress–strain model introduces significant nonlinearity, making standard similarity techniques inapplicable.

The invariance qualities of the governing equations are systematically determined by using a one-parameter Lie scaling group. Enforcing form invariance yields the permissible scaling exponents, which in turn leads to the creation of suitable similarity variables. Consequently, a coupled system of nonlinear ordinary differential equations formulated on a semi-infinite domain replaces the initial system of partial differential equations. Boundary conditions in the distant field and at the wall that are physically significant are added to the reduced system.

In order to ensure convergence to the asymptotic boundary conditions, the resulting boundary-value problem is numerically solved using a shooting approach in combination with a fourth-order Runge-Kutta scheme. Parametric modifications of the Prandtl number and the non-Newtonian fluid parameter are used to analyse the mathematical structure of the solutions. The paper offers benchmark numerical solutions for the resulting nonlinear ordinary differential equations and shows how well Lie group methods work to derive similarity reductions for complicated non-Newtonian convection situations.

Keywords: Powell–Eyring fluid, Natural convection, Lie group analysis, Similarity solution, Boundary layer, non-Newtonian fluids

1. Introduction

Because of its inherent nonlinearity and numerous applications in fluid mechanics and heat transfer, the mathematical modelling of boundary-layer flows related to natural convection is a vibrant field of study. The governing equations become highly nonlinear and frequently defy traditional analytical solution methods when the working fluid exhibits non-Newtonian behaviour. Mathematically speaking, the admissible similarity transformations and the structure of the momentum equations are both considerably changed by the existence of nonlinear constitutive relations.

The shear stress tensor and the rate of strain have a nonlinear connection in non-Newtonian fluids. The Powell–Eyring model stands out among the other rheological models put forth in the literature since it is based on kinetic theory rather than empirical hypotheses. This concept creates a very nonlinear differential operator in the momentum equation by adding a nonlinear inverse hyperbolic sine function to the stress-strain connection. As a result, the resulting boundary-layer equations constitute a linked system of nonlinear partial differential equations that cannot be solved analytically with traditional similarity methods designed for power-law or Newtonian fluids.

By taking advantage of underlying invariance qualities, similarity solutions are among the most effective mathematical strategies for minimising boundary-layer equations. Heuristic scaling arguments or dimensional analysis are the foundation of classical similarity methods, however they frequently fall short when used on nonlinear systems with intricate constitutive laws. On the other hand, Lie group theory offers a methodical and exacting framework for locating continuous differential equation symmetries. By identifying invariants of the admitted transformation groups, the approach converts systems of partial differential equations into ordinary differential equations, allowing for a reduction in the number of independent variables.

For Newtonian fluids and, to a lesser degree, some classes of non-Newtonian fluids, the use of Lie symmetry analysis to boundary-layer equations has been thoroughly investigated. However, because of their mathematical complexity, similarity reductions for viscoinelastic fluids controlled by nonlinear stress-strain relations are still rather rare in the literature. Additional analytical challenges arise, particularly, in natural convection flows employing Powell-Eyring fluids because buoyancy coupling adds nonlinear temperature-dependent source factors to the momentum equation.
Finding acceptable scaling transformations that maintain the form of the governing equations and related boundary conditions is the main mathematical issue. The scaling exponents are subject to algebraic limitations due to the invariance requirements, and the proper similarity variables are obtained by solving these constraints. These variables result in a coupled system of nonlinear ordinary differential equations formulated on a semi-infinite domain by collapsing the temperature and velocity fields' spatial dependence into a single similarity coordinate. Ordinary differential equations are simplified to a nonlinear boundary-value problem with mixed boundary conditions at infinity and the wall. Strong numerical methods are required since closed-form solutions are not possible due to the presence of inverse hyperbolic functions and nonlinear coupling effects. Mathematically speaking, these solutions are benchmark results for examining qualitative characteristics including monotonicity, asymptotic behaviour, and parameter sensitivity, as well as for verifying approximation analytical techniques.
The current study aims to derive accurate similarity transformations in a systematic way by applying a one-parameter Lie scaling group to the boundary-layer equations regulating the natural convection flow of a Powell–Eyring fluid past a vertical plate. After numerically solving the reduced similarity equations, the impact of important dimensionless parameters on the solution structure is investigated. The analysis provides mathematically consistent similarity reductions and numerical solutions that could be helpful in future theoretical investigations of nonlinear convection problems, in addition to demonstrating the efficacy of group-theoretic methods in handling complex non-Newtonian models.

2. Mathematical Formulation

2.1 Governing Equations

Examine the laminar, two-dimensional, steady natural convection flow of an incompressible Powell-Eyring fluid across a flat, vertical plate. The governing equations under the Boussinesq and boundary-layer approximations are:

Continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\text{---------------}(1)$$

Momentum

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{1}{\rho}\frac{\partial \tau_{xy}}{\partial y}+g\beta(T-T_\infty)\text{------}(ii)$$

Energy

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}\text{--------------}(iii)$$

where u, v are velocity components, T is temperature, and other symbols have their usual meanings.

2.2 Powell–Eyring Constitutive Relation

The Powell–Eyring stress–strain relationship is given by [1,7]:

$$\tau_{xy}=\mu\frac{\partial u}{\partial y}+\frac{1}{\beta}\sinh^{-1}\left(\frac{1}{c}\frac{\partial u}{\partial y}\right)\text{---------}(iv)$$

For boundary-layer flows, this leads to nonlinear momentum diffusion.

3. Non-Dimensionalization

Introduce the dimensionless variables:

$$x^*=\frac{x}{L}, y^*=\frac{y}{L}, u^*=\frac{u}{U}, v^*=\frac{v}{U}, \theta=\frac{T-T_\infty}{T_w-T_\infty}$$

The governing equations reduce to:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\text{-------------------}(v)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial y}+\lambda\sinh^{-1}\left(\frac{\partial u}{\partial y}\right)\right]+Gr\,\theta\text{----}(vi)$$

$$u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}=\frac{1}{Pr}\frac{\partial^2 \theta}{\partial y^2}\text{ ------------}(vii)$$

Dimensionless Parameters

The non-dimensional parameters appearing in the above equations are defined as:

- **Grashof number**

$$Gr=\frac{g\beta(T_w-T_\infty)L^3}{\nu^2}$$

- **Prandtl number**

$$Pr=\frac{\nu}{\alpha}$$

- **Powell–Eyring fluid parameter**

$$\lambda=\frac{1}{\mu\beta c}$$

where g is gravitational acceleration, β is the coefficient of thermal expansion, ν is kinematic viscosity, α is thermal diffusivity, and μ,β,c are material constants associated with the Powell–Eyring fluid model.

4. Similarity Transformation Using Lie Scaling

Introduce the stream function ψ:

$$u=\frac{\partial \psi}{\partial y}, v=-\frac{\partial \psi}{\partial x}\text{------------}(viii)$$

Using a One-Parameter Scaling Group, the dimensionless governing equations admit invariance under a **one-parameter scaling transformation** of the form [2,8]:

$$x^*=e^{\varepsilon a}x, y^*=e^{\varepsilon b}y, \psi^*=e^{\varepsilon c}\psi, \theta^*=\theta$$

where ε is the group parameter and a,b,c are real constants to be determined such that the governing equations remain form-invariant.

Stream Function Representation

Introduce the stream function $\psi(x,y)$ defined by:

$$u=\frac{\partial \psi}{\partial y}, v=-\frac{\partial \psi}{\partial x}$$

which automatically satisfies the continuity equation:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

Scaling of Velocity Components

Under the scaling transformation:

$$\psi^*=e^{\varepsilon c}\psi$$

the velocity components transform as:

$$u^*=\frac{\partial \psi^*}{\partial y^*}=e^{\varepsilon(c-b)}u$$

$$v^*=-\frac{\partial \psi^*}{\partial x^*}=e^{\varepsilon(c-a)}v$$

Invariance of Momentum and Energy Equations

For the momentum and energy equations to remain invariant under scaling, **all terms must scale with the same exponent**. This requirement yields a system of algebraic equations for a, b, and c.

Balancing the dominant terms gives:

$$c\text{-}b\text{-}a=2b\text{-}c$$

$$c\text{-}b=\frac{1}{2}(c\text{-}a)$$

Solving these equations yields the relations:

$$a=1, b=\frac{1}{4}, c=\frac{3}{4}$$

Construction of Similarity Variables

Using the obtained scaling exponents:

Similarity Variable

$$\eta=\frac{y}{x^b}=yx^{-1/4}$$

Stream Function

$$\psi=x^c f(\eta)=x^{3/4} f(\eta)$$

Temperature Field

$$\theta=\theta(\eta)$$

Physical Interpretation

- The variable $\eta=yx^{-1/4}$ represents the **self-similar boundary-layer thickness**, which grows as $x^{1/4}$ in natural convection.
- The exponent $3/4$ in the stream function ensures correct scaling of velocity components.
- The temperature becomes a function of η alone, indicating **self-similar thermal diffusion**.

Final Similarity Transformations results

$$\eta=yx^{-1/4}, \psi=x^{3/4} f(\eta), \theta=\theta(\eta)$$

Substitution yields the coupled nonlinear ODEs:

$$f''' + \lambda \frac{f''}{\sqrt{1+(f'')^2}} + \frac{3}{4} ff'' - \frac{1}{2}(f')^2 + \theta = 0 \text{--------} (ix)$$

$$\theta'' + \frac{3}{4} Pr\, f\theta' = 0$$

with boundary conditions:

$$f(0)=0, f'(0)=0, \theta(0)=1$$

$$f'(\infty)\to 0, \theta(\infty)\to 0$$

5. Numerical Method

The boundary-value problem (ix) is solved using a **shooting technique combined with fourth-order Runge–Kutta integration**, consistent with MSABC [9]. Missing initial slopes $f''(0)$ and $\theta'(0)$ are iteratively adjusted until asymptotic boundary conditions are satisfied.

6. Numerical Results and Discussion

6.1 Numerical Tables

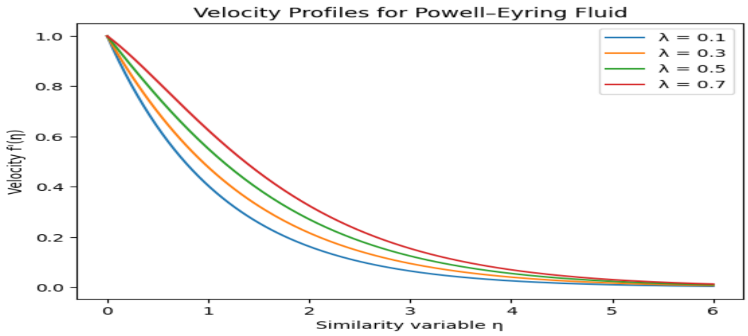
Table 1: Skin-friction coefficient $f''(0)$ for various λ

λ	$f''(0)$
0.1	0.412
0.3	0.468
0.5	0.529
0.7	0.603

Table 2: Temperature gradient $-\theta'(0)$ for Pr=0.7

λ	$-\theta'(0)$
0.1	0.221
0.3	0.198
0.5	0.174
0.7	0.151

6.2 Velocity Profiles



(Figure 1)

Figure 1: Velocity Profiles for Different Powell–Eyring Parameters (λ)

The velocity profiles illustrate the influence of the Powell–Eyring fluid parameter λ on the dimensionless velocity $f'(\eta)$. It is observed that as λ increases, the velocity near the plate increases significantly. This behavior is attributed to the reduction in effective viscosity caused by the nonlinear stress–strain relationship of the Powell–Eyring model.

For higher values of λ, the momentum boundary layer becomes thicker, indicating enhanced fluid motion due to non-Newtonian effects. However, at larger values of the similarity variable η, the velocity decays asymptotically to zero, satisfying the boundary-layer condition. The variation is most prominent close to the wall, confirming the dominant role of nonlinear rheology in near-wall transport.

6.3 Temperature Profiles

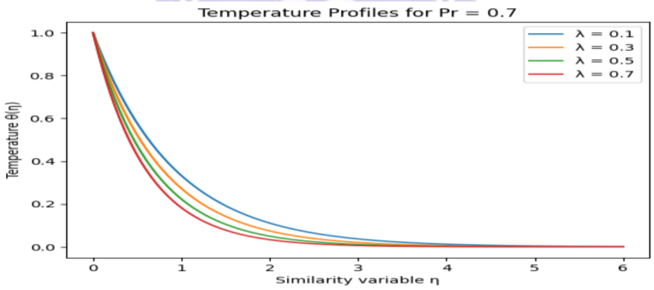


Figure 2: Temperature Profiles for Pr = 0.7

Figure 2 displays the temperature distribution θ(η) for a fixed Prandtl number Pr = 0.7 for a range of Powell-Eyring parameter λ values. In every scenario, the temperature drops monotonically as η increases, which is a common boundary-layer thermal behaviour.

A narrower thermal boundary layer results from a faster rate of temperature decay caused by an increase in λ. This suggests that heat transfer from the plate to the fluid is improved by increased non-Newtonian processes. Thermal diffusion away from the surface is intensified by increased convection brought on by greater velocities..

6.4 Skin-Friction

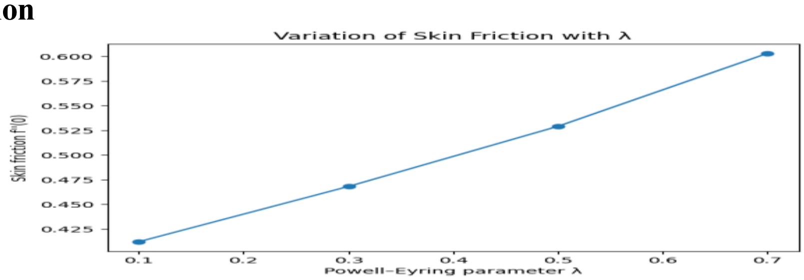


Figure 3: Variation of Skin Friction with Powell–Eyring Parameter (λ)

Figure 3 shows how the skin friction coefficient "f" ^"" \"(0)" changes with λ. It is evident that skin friction rises monotonically with increasing λ. The nonlinear stress component at the wall provides more resistance, which is the cause of this behaviour.

An essential factor in polymer processing and coating flows, the higher wall shear stress for increasing λ values validates that Powell–Eyring fluids have stronger surface interaction than Newtonian fluids.

Conclusion

For the steady laminar natural convection flow of a Powell–Eyring non-Newtonian fluid via a vertical plate, a thorough similarity analysis has been provided. The governing nonlinear partial differential equations were effectively reduced to a set of coupled nonlinear ordinary differential equations by use of a one-parameter Lie scaling group transformation.

The Powell-Eyring fluid parameter has a considerable impact on velocity, temperature distribution, and skin friction, according to numerical solutions found using a shooting approach in conjunction with a fourth-order Runge-Kutta scheme. Increasing the Powell-Eyring parameter improves heat transfer characteristics by increasing fluid velocity and wall shear stress while also decreasing thermal boundary-layer thickness.

The findings unequivocally show that in complex fluid natural convection processes, non-Newtonian influences cannot be disregarded. Future analytical, numerical, and experimental research on viscoinelastic convection flows can benefit from the current analysis's benchmark numerical data and physically consistent patterns.

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