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Interplay Between Differential Geometry and Topology: Recent Trends in Theoretical and Applied Research

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Abstract

This paper reviews the interrelated theories of differential geometry and topology, which play a key role in the continuously emerging research areas of modern mathematics, data science, machine learning, biology and theoretical physics. While differential geometry studies the curvature and local structures of smooth surfaces, topology focuses on properties that remain unchanged despite continuous changes, such as loops, holes and connectivity. The paper mentions technical tools such as topological data analysis (TDA), continuous symmetry, Ricci curvature, mapper algorithm, graph neural networks and Ricci flow, which are helpful in understanding high-dimensional and complex data structures. The study also highlights that systems such as AlphaFold2, which use these methods in protein structure prediction, represent revolutionary advances in science. Similarly, the utility of these theories is also demonstrated in areas such as single-cell data analysis, topological phases, relativity theory and quantum geometry. In conclusion, this study proves that the combination of differential geometry and topology not only provides a mathematically consistent and sophisticated framework but also helps promote innovation and explainability in machine learning and science.

Keywords - Differential Geometry, Topological Data Analysis (TDA), Ricci Curvature and Flow, High-Dimensional Data Structures.

I. Introduction

Differential geometry is a branch of mathematics that studies properties of curvature and smooth surfaces, while topology focuses on qualitative features of space that remain unchanged after constant change (such as stretching or folding). One geometry alone is not enough to understand the structures of nature in this context, asymmetry developed, so that abstract properties of solid geometry, such as continuity and coincidence became prominent the convergence of these two branches gives various frameworks in modern mathematical sciences.

The interrelationship between differential geometry and asymmetry in current research opens up new dimensions in several areas. For example, in modern machine learning and data science, data sets are treated as (multiple) samples on a high-dimensional surface; Understanding the "shape" of these data helps identify underlying patterns. In this respect, techniques such as topological deep learning (TDL) have made it possible to analyse complex, high-dimensional data by incorporating principles of topology into deep neural networks The central mechanism is permanent symmetry, which combines abstract heterogeneity and complex geometry using filtering

Besides traditional mathematical methods (such as math, linear algebra, etc.) in machine learning and data science, advanced concepts such as geometry, heterogeneity, and group theory continue to play a growing role in modern research They are included. For example, morphological features present in the data (such as loops, holes in the data, etc.) can be identified using permanent symmetry techniques, providing more meaningful features in classification, clustering, and other machine learning tasks

The use of differential geometry and anisotropy in image analysis and computer vision is growing rapidly. An image can be viewed as a collection of high-dimensional surface patterns. Differential geometry analyses the smoothness and curvature of image surfaces, while dissimilarity helps identify qualitative structures of shapes (such as continuous elements and connectivity of shapes) From this perspective, topological data analysis (such as Mapper algorithm) is used to uncover hidden patterns and features in image data.

This interaction is quite intense in theoretical physics. Differential geometry in general relativity defines equations of curvature of spacetime structure, yet inequalities in quantum field theory and string theory, such as fibre fold topology and classification of topological phases, are useful for understanding complex states of matter and the universe plays an important role

These concepts are also proving extremely useful in computational biology. For example, static symmetry is used to understand high-dimensional structures in single-cell biological data. A recent review shows that this method can be used to detect rare cell clusters, transitional states, and branching differentiation pathways that traditional analysis hides as well as integrating deep learning with geometric and topological approaches in predicting protein structure (e.g. AlphaFold2).

Prime examples of these interdisciplinary advances include systems like AlphaFold2, which uniquely predicted protein structures with deep learning, won a 2024 Nobel Prize in chemistry and revolutionary advances in disease diagnostics, drug design and materials science due to topographic data analysis based computational tools

Thus, integrating differential geometry and anisotropy not only enriches mathematical research but provides a solid foundation for innovation in modern machine learning, data science, sensory-based image analysis, computational biology, and theoretical physics We will highlight trends and challenges prevailing in the regions in depth

II. Review of Related Literature

Recent years have seen a significant convergence of differential geometry and topology, both in theoretical mathematics and applied computing fields. Essentially, topology explores qualitative aspects of space, such as connectedness and continuum, while differential geometry adds the machinery of curvature, geodesics, and smooth structures. This interrelation has facilitated the emergence of new techniques in machine learning, data science, and mathematical physics.

Topological data analysis (TDA) has become a powerful tool for extracting meaningful features from high-dimensional data. Chazal and Michel (2021) provide a comprehensive introduction to TDA, demonstrating its practical value in data science. A notable contribution is by Adams et al. (2017), who introduced Persistence Images, a stable and vectorized representation of Persistent Homology, which helps connect topology to machine learning algorithms.

The development of geometric deep learning was based on the idea that most real-world data does not have a Euclidean structure, but is found in graphs, manifolds, or other complex topological structures. Bronstein et al. (2017) pointed out that traditional machine learning techniques are not sufficient to understand the structure of such data and perform accurate computations on it. They argued that geometric and topological approaches are needed to identify the deep patterns that exist when data is contained in nonlinear structures such as graphs or manifolds. Extending this idea further, Kwit et al. (2017) in their study incorporated topological signatures (such as properties derived from permanent symmetries) into deep neural networks, thereby improving the pattern recognition capabilities of the models. They showed how these topological features not only make models more accurate but also provide interpretability a major goal of today’s AI research. This research shows that if topology is incorporated into machine learning models, they become more stable, more flexible, and better able to understand complex structures. This whole direction is part of the research that is moving towards making modern AI more transparent and robust.

In biomedical applications, the innovative use of topological data analysis (TDA) has opened new avenues for understanding complex biological data. A landmark study by Nicolaou et al. (2011) demonstrated the potential of this approach by applying TDA to breast cancer gene expression data. Through their analysis, they identified a previously unknown subgroup of breast cancer patients that displayed a unique mutation profile and significantly better survival rates than other groups. Unlike traditional statistical techniques that often rely on linear

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assumptions or predefined classifications, the topological method used in this study enabled the exploration of the shape and structure of the high-dimensional data space. By leveraging tools such as Persistent Homology and the Mapper algorithm, the researchers were able to detect subtle patterns and nonlinear relationships within the data that might otherwise have been overlooked. This discovery not only had profound clinical implications, suggesting the existence of more subtle biological subtypes in breast cancer, but it was also a turning point in the field, demonstrating how topological descriptors can reveal hidden biological insights. It confirmed the idea that topology-based approaches are not only mathematically beautiful but also practically powerful in uncovering complex, meaningful patterns in biomedical datasets, where traditional methods may fall short.

A central theory of geometric analysis, Ricci Flow, originally developed to represent the evolution of curvature in Riemannian geometry, has now been effectively applied in computational contexts as well. Ni and colleagues (2019) used this theory to detect communities in complex networks. They demonstrated how geometric evolution equations such as Ricci Flow can help identify underlying modular structures in data. Extending this theory further, Baptista and colleagues (2024) attempted to explain the training dynamics of deep learning through Ricci Flow. They viewed layer-wise transformations as a type of geometric evolution, equivalent to Hamilton's Ricci Flow. This approach makes it clear that the transformations that occur in deep learning are not merely statistical but also have a deeper geometric meaning. This idea is extremely important in the context of explaining artificial intelligence (AI), as it presents the training process as an evolving geometric system, not just a black-box. Thus, the use of Ricci Flow in the computational domain not only provides a better understanding of data structure but is also a major step towards making AI systems more transparent and scientifically explainable.

From the physics point of view, Qi and Zhang (2011) presented an important review on topological insulators and superconductors, explaining how topological invariants such as the Chern number or the Z2 invariant determine the structure of quantum states and their behaviour. They showed that phenomena such as surface states, electrical conductivity, and the quantum Hall effect can be understood more deeply by taking the topological properties of a material into account, opening up new avenues in areas such as quantum computing and spintronics. In the same context, Mathai and Thiang (2017) analysed the differential topology of semimetals and showed that the quantum properties of materials such as Dirac and Weyl semimetals can be better understood using mathematical frameworks such as fibre bundles, cohomology, and combinatoriality. They showed that topological models explain not only static materials but also complex quantum states arising under changing conditions and external influences. Both of these studies proved that topological and differential geometric approaches have become integral tools in modern theoretical physics, contributing profoundly to the discovery and analysis of new states of matter.

Apart from theoretical and physical sciences, topology has also made significant contributions in the field of data visualization and interpretation. The Mapper algorithm presented by Lum et al. (2013) has proved to be a revolutionary technique in this direction. This algorithm produces a topological summary of a complex and high-dimensional data set, through which the shapes, structures and patterns hidden in the data can be displayed in a visual form. While numerical analysis is dominant in traditional statistical techniques, the Mapper approach is based on the topological structure, which has the ability to simultaneously incorporate global and local properties of the data. This technique is particularly useful when the data is highly complex, noisy or nonlinear, as is often seen in bioinformatics, cancer research, social network analysis and financial statistics. For example, identifying subgroups in genomic data, classifying disease types, or discovering influential sub-communities in social media networks can be done with the help of the Mapper algorithm. Its graph-like output (nodes and edges) allows users to not only see the gist of the data, but also interpret and analyse it easily. Thus, Mapper connects algorithmic diversity to the world of behavioural and visual analytics, making it an extremely useful tool for modern data science and machine learning.

It is evident from these studies that the topological invariants of curvature and skewness of differential geometry together provide a powerful means to deeply understand the structure, meaning, and variability of various data sets. Curvature measures the local geometric properties of data such as inclination, surface smoothness, and distance, while topological invariants such as holes, loops, and connectivity capture global structural properties that remain unchanged despite continuous change. When the two are integrated, such as by adding curvature-based features to graph neural networks or adding topological data analysis (TDA) to deep learning, both the interpretability and functionality of models are significantly improved. This coordinated approach not only makes AI more transparent and reliable, but also provides new insights and consistency in scientific modelling. Such hybrid frameworks will pave the way for interpretable AI, robust scientific analysis, and advanced theoretical research in the future.

III. Theoretical Foundations

Differential geometry and topology are two rich fields of mathematics that help understand the fundamental concepts of shape, space, and structure and when combined present a revolutionary approach in modern mathematics and applied research. Differential geometry mainly studies the behaviour of smooth manifolds, curvature, geodesic, and metric structures, while topology investigates the properties of shapes that remain unchanged during continuous transformations, such as connectedness and number of holes. The convergence of these two fields through the Gauss-Bonnet theorem and the Atiyah-Singer index theorem shows that local geometric properties are deeply connected to global topological properties. In recent years, this interdisciplinary approach has expanded to various fields, with innovations such as topological data analysis (TDA), Persistence Images (Adams et al., 2017) and the Mapper algorithm (Lum et al., 2013) for applying topological features to machine learning algorithms. Differential geometric theories such as Ricci flow have also been used in network science for community detection and network ordering (Ni et al., 2019). Baptista et al. (2024) connected it to the layer-by-layer transformation of deep learning, which was seen as analogous to Hamiltonian geometry. On the other hand, these concepts are also influential in areas such as topological matter and quantum insulators, as described by Qi & Zhang (2011) and Mathai & Thiang (2017). Topological invariants such as Euler characteristic and Betti number are being used today in cancer diagnosis (Nicolaou et al., 2011), social network analysis, and biological data processing. This interaction is not only important theoretically, but also strengthens performance, interpretability, and generalization capability in practical areas. Thus, the integrated approach of differential geometry and topology not only makes the mathematical framework more widely understandable, but also continually enhances its relevance and utility in a variety of scientific fields.

IV. Thematic Domains of Interplay A. Gauss-Bonnet and Index Theorems:

The Gauss-Bonnet theorem is one of the most important discoveries in the history of mathematics, linking local curvature to global topological properties. This theorem applies uniquely to a finite, oriented two-dimensional Riemannian manifold and states that the total Gauss curvature of a surface is equal to the Euler characteristic of that surface. In simple terms, when we sum all the curvatures of a surface, we get the sum of the number of "thicknesses" or "holes" of that surface. The importance of this theorem is not just limited to its theoretical beauty, but it has also proven to be extremely useful in classical mathematics and modern data science, especially when data is treated as a manifold. For example, this theorem is particularly useful in topological data analysis (TDA) as it helps to understand the global structure based on local measurements (such as curvature). (2017) brought this concept into practice, where the structure of high-precision data is revealed through high-level computations. Similarly, the

Mapper algorithm proposed by Lum et al. (2013) is also a tool for students of Gauss-Bonnet theory to understand the concepts of statistics. In contrast, the Atiyah-Singer index theorem is a deep and complicated theorem that connects global topological invariants (such as Euler characteristic, signature, or KO-dimension) to properties of local differential equations. This theorem connects the evenness (or oddness) of solutions of a differential equation to properties of the topology of arithmetic. It is particularly useful in geometric crystals, quantum field theory, electronic theory, etc. and in many other areas of contemporary mathematics.

B. Characteristic Classes:

Characteristic classes show a deep connection between differential geometry and topological mathematics, transforming the geometric structure of a vector bundle into global topological properties. These classes, such as the Chern, Pontryagin and Stiefel-Whitney classes, are related to the curvature of bundles and are based on global differential forms derived from their local geometry. A distinctive feature of these classes is that they are topological invariant, i.e., the shape of the surface remains unchanged even when stretched or bent, as long as no topological changes occur. Models such as the Chern Simons theory in mathematical physics and theorems such as the Atiyah Singer index theorem are based on these classes, showing a deep connection between global and local computation. In contemporary research, especially in topological data analysis (TDA), machine learning and complex network analysis, these classes are being used to gain a global understanding of complex structures. They are helpful in uncovering invisible features of biological networks, social graphs and high-dimensional data spaces. Thus, symbolic classes are not only a pure mathematical concept, but also play a fundamental role in a variety of modern scientific and technological applications, where they serve to relate the local geometry of vector bundles to global data features.

C. Ricci Flow and Geometric Evolution:

Ricci flow and the theory of geometric evolution serve as an important bridge between mathematics and applied science, where the geometry of a manifold evolves over time. Ricci flow, first introduced by Richard S. Hamilton, is a method in which the metric of a manifold is changed over time such that the curvature gradually “fetters”, making the structure more regular and proportional. The process is considered relatively similar to the heat equation, but applied to curvature. Ricci flow has had a wide influence not only in mathematical fields, such as the solution of the Poincare conjecture (through the work of Grigori Perelman), but also in modern fields such as machine learning and data science. In recent years, Baptista et al. (2024) have studied in detail how the training of deep neural networks can be viewed as a Ricci flow, where changes in layers are sequentially similar to the arithmetic mean. From this perspective, controlling the curvature of a model can improve its learning ability and stability. Ricci flow can not only be used to smooth arithmetic, but it has also been used as an effective tool in the analysis of graph structures, biological networks, and high-dimensional data. This concept of changes in curvature over time is also useful to model structural changes in data, such as community formation or network disintegration. Thus, Ricci flow and geometric evolution provide cross-branching insights into many aspects of modern mathematics and science.

D. Topological Quantum Field Theory (TQFT):

Topological quantum field theory (TQFT) is a modern and interdisciplinary approach that model’s quantum metrics through topological quantum field theory, where complex physical phenomena can be understood only by the "morphology" of any space or element, not by their scale or shape. The basic idea of TQFT is that the behaviour of a quantum field theory depends only on the topological structure of the space it defines, such as its orientability, holes or symmetries. In this process, discrete topological spaces are modelled as geometric theories, where the geometry is interpreted through architectural tools such as differential forms, adjunctions and holonomy. Mathematicians and physicists such as Atiyah and Witten played a leading role in the development of this theory, where it was proven that TQFT not only provides the basis for understanding theories, but also helps explain mysterious hypotheses of physics such as quantum mechanical theory, brane fields (braiding fields) and quantum entanglement. In addition, TQFT has also found application in modern data analysis and quantum classification, where the global topology of any network has traditionally been used. This view of TQFT represents the point where topology and differential symmetry come together to provide expressions of mysterious physical claims, and it proves to be a powerful example for interdisciplinary research.

E. General Relativity and Quantum Geometry:

Einstein's general theory of relativity describes spacetime as a smooth, curved manifold, in which the effect of gravity is expressed as curvature produced by mass and energy. The mathematical basis of this model is Riemann geometry and in particular Ricci curvature, which combine energy-dynamics and geometric structure. But when we reach the microscopic level, i.e. the quantum scale (Planck scale), this model proves inadequate and the need for quantum gravity arises, where spacetime is no longer continuous but becomes a realm of probabilistic and dynamic topological structures. In such a perspective, alternative theories emerge, such as topological quantum field theory (TQFT), loop quantum gravity (LQG), string theory and non-commutative geometry. Using mathematical tools such as Ricci flow, holonomy and fibre bundles, the geometric progression of spacetime over time has been modelled, thereby establishing a deep interrelationship between not only physics but also two major branches of mathematics, differential geometry and topology. This interrelationship opens new doors in modern science for understanding the structure of spacetime, quantum explosions, gravitational waves, and the fundamental nature of the universe.

V. Applications in Computational and Applied Domains A. Geometric Deep Learning:

The concept of geometric deep learning became more relevant when it became clear that most real-world data do not exist in traditional Euclidean spaces, but in complex structures such as curvilinear manifolds or graphs Bronstein et al (2017) in this field. (2017). These techniques mainly use curvature-informed messaging, which takes the efficiency of graph neural networks (GNNs) to new heights. This led Adams et al. Innovations such as Persistence Images proposed by (2017) and Topological Data Analysis (TDA) presented by Chazal & Michel (2021) also define the field, where many hidden structural properties can be exposed and translated into machine learning algorithms How modelling itself is a geometric inversion Overall, how differential geometry and algebraic topology provide new directions for structural sophistication, interpretability, and predictability in artificial intelligence It's a perfect example of convergence

B. Topological Data Analysis:

Topological Data Analysis (TDA) is a modern and powerful mathematical method that helps to identify and analyse hidden structures in high-dimensional data This method uses topological invariants such as Betty numbers and Euler characteristics to understand the complexity of the data. can understand the shape of data and its combination through measurements, thereby revealing patterns that remain hidden from traditional statistical and geometric methods The introductory work on TDA presented by Chazal and Michel (2021) shows how this technique has become a practical tool for data scientists. The concept of sustainability models developed by Adams et al. (2017) represents continuous symmetry as a linear vector, which allows for easy integration of topological features into machine learning algorithms. It also attenuates noise, e.g., Nicolaou et al. (2011) used topological analysis to identify a specific subgroup of breast cancer patients who had a distinct mutation profile and good survival proving that topological techniques can be more sensitive and practical than traditional biostatistical methods. Therefore, TDA is not just a technical tool of modern data science but emerged as an innovative tool to increase structural understanding, interpretability

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and depth of findings, directly linking sophisticated tools of topology and differential geometry with experimental and applied sciences

C. Computer Graphics:

In computer graphics, discrete differential geometry serves as a fundamental framework to precisely model, visualize, and manipulate digital geometric objects, especially polygon grids that represent surfaces in computing environments unlike classical differential geometry, which deals with smooth, continuous forms Preservation of geometric properties and enables efficient computation (Crane et al., 2013) These properties form the basis of many essential processes in computer graphics, including realistic surface rendering through shading techniques such as Fong shading and bump mapping, as well as discrete parameterization (Gu & Yau, 2008) Also, texture mapping benefits from such discrete frameworks, as do shape distortion, animation, and mesh smoothing applications-often inspired by Ricci's flow-of digital models allowing coherent physically plausible changes (Jin et al., 2008), these capabilities biomedical visualization and digital reconstruction etc. is especially valuable in fields, where accuracy and realism are important Thus discrete differential geometry not only bridges the gap between abstract mathematical theory and applied visual computation but also fosters innovation in rendering, simulation, and interactive 3D design (Wardetzky et al., 2007).

D. Biological and Material Sciences:

. In biological sciences, the integration of multiple learning and topological invariance has emerged as a transformative approach to model and analyse the complexity of biological systems and molecular structures. For example, topographic data analysis (TDA) in genomics and proteomics allows researchers to consistently detect homogeneous features – such as loops or gaps – producing high-dimensional biological data sets that remain unchanged under continuous changes (Adams et al., 2017). These features often correspond to biologically meaningful patterns, such as gene expression sets or protein-protein interaction modules, which are difficult to identify using linear or Euclidean techniques. Using Ricci curvature and other differential geometric tools further enhances our ability to model molecular stability, folding patterns, and energy landscapes. Studies such as Nicolaou, Levin, and Carlson (2011) have demonstrated the real-world utility of these methods for example, by identifying breast cancer subtypes with specific mutational profiles and improving survival rates based solely on topological features extracted from gene expression data. Thus, the combination of differential geometry and topology in biological physics provides not only theoretical beauty

VI. Open Challenges and Future Directions

Unification of Discrete and Continuous Frameworks

Integrating the curvature and topology of discrete and continuous forms is a fundamental challenge in contemporary mathematics and computational science. Discrete structures, such as graphs and simple complexes, inherently lack differentiating properties, making it impossible to directly define curvature and topological properties, and require resorting to combinatorial choices Approximates the Ritchie curvature of the surface of the base but this approach is often applicable to ideal geometric graphs, still not a fully integrated framework working for common data sets. For example, it is still an open problem to discretely define the generalized Gauss Bonnet theorem or the Atiyah–Singer index. Similarly, integrating continuous topologies (such as de Rahm cosymmetry) and discrete topologies (such as simple symmetry) is especially difficult when data exist in mixed continuous discrete forms It provides a more robust and interpretable structure for modern artificial intelligence and data analytics

Algorithmic and Scalability Issues

Many differential geometry and topology-based methods are extremely complex and resource-intensive from an algorithmic and scalability perspective. For example, computing the Olivier-Ricci curvature of a graph requires solving an optimal transportation problem on each edge, which can have a complexity of up to $O(|E| \cdot m^3)$ in the worst case, where $|E|$ is the number of edges in the graph and m denotes the maximum degree. Even the relatively simple Forman curvature has a complexity of O (|E| · m) because it involves computing local structures (motifs). Similarly, techniques such as persistent homology also impose a heavy computational burden on high-dimensional and large data sets - for example the Vietoris-Rips complex based on n points. When these methods are applied to large-scale artificial intelligence projects, data science tasks, or high-dimensional point clouds, their computational cost often becomes a major bottleneck. Although faster algorithms (such as equational methods or parallel processing) are being developed, scalability is still a major challenge. For example, some specific graph libraries such as the Graph Neural Network Framework are able to handle large data, but adding the calculation of curvature or high-dimensional topological features to them may adversely affect their efficiency. Therefore, developing algorithms that can efficiently calculate these topological and geometric descriptors on large data sets remains a major open research problem.

Hybrid Geometric–Topological Frameworks

Hybrid models combining differential geometry and disparity are considered a very promising direction in today's artificial intelligence (AI) and data science fields, as these models combine local geometric structure and global topological stability. Topological invariants such as permanent homology are highly stable to noise, while curvature (such as Ricci or Forman curvature) helps to highlight subtle local features of the data. Integrating the two has significantly improved the accuracy and interpretability of models; for example, the Persistence Image technique presented by Adams and colleagues (2017) converts homology features into machine learning-friendly vectors, while adding curvature-based features to new graph neural networks has improved classification results and reduced the problem of over-homology. The beauty of these hybrid methods is that they not only work efficiently, but also make model decisions more transparent and interpretable, increasing both the scientific use and social credibility of AI systems. As a result, the combination of topological techniques such as Betti numbers and Mappers with geometric tools such as Ricci flow and Riemannian metrics is emerging as a very active and innovation-driven research area, aimed at building more robust, interpretable, and principled AI systems.

Dynamic and Evolving Structures

Many real-life problems today involve structures that change over time or contain high-order interactions (e.g., multi-way connectivity), which existing differential geometric and heterogeneous tools can only partially handle. For the analysis of dynamic data such as time-varying networks or moving point clouds, some new techniques have been developed – such as the "vineyard", the "Crocker plot" or the "Crocker stack" – that abstractly represent time-dependent changes of the topology. Nevertheless, the theory of analysing dynamic geometric structures, such as flowing manifolds or network flows, is still in a developing stage. The recently developed concept of "dynamic Ricci curvature" (i.e. time-dependent Olivier-Ricci curvature that transmits information at different levels) is considered a promising step in this field. For example, Tannenbaum and colleagues (2024) showed that if the curvature is measured as a time-dependent function, it can reveal multi-level community structure in genomic networks. Similarly, hypergraph structures in which a single connection can connect multiple nodes present a new theoretical challenge. According to Yang and colleagues (2024), generalizing curvature to hypergraphs can be useful for identifying high-level communities, but it is either too expensive (e.g., multi-marginal transport-based curvature models) or loses descriptive power if a simple combinatorial formulation is used. It is therefore clear that the concepts of differential geometry (e.g., curvature, metrics) and asymmetry invariance need to be extended to structures such as time-varying manifolds, hypergraphs, and streaming data, which requires the development of new theories and scalable algorithms.

Ethical and Societal Considerations

As advanced mathematical concepts such as curvature and topology are increasingly used in artificial intelligence (AI), it is also becoming important to consider their ethical and social dimensions. On the one hand, these techniques can promote fairness and transparency. For example, Singh and colleagues (2024) found that topological data analysis (TDA) can reduce bias in medical image classification by identifying subtle structural differences; Srinivasan and Chander (2023) showed that visualizations based on persistence-symmetry can help analyse biases (e.g., age discrimination) in datasets. But on the other hand, this mathematical complexity can make the decision-making process so vague that it becomes difficult to identify the underlying biases within the model. If there are differences in the size or contact patterns of a group, such features may inadvertently encode sensitive attributes (e.g., race, gender, age). As a solution, approaches such as interpretable models, "audit" analysis, and topological visualizations should be incorporated to make decision-making more clear, equitable, and accountable (Singh et al., 2024; Srinivasan and Chander, 2023).

VII. Conclusion

In contemporary mathematics, the theoretical unification of dual geometry and topology has provided a deep pedagogical foundation, where local curvature and metric structures of smooth manifolds are closely linked to global topology indices. The Gauss–Bonnet theorem shows that the total Gauss curvature of a Riemannian surface is its Euler characteristic, establishing an explicit connection between local curvature and global topology. Similarly, the Atiyah–Singer index theorem connects the computation of solutions of local differential equations to global topology properties, thereby unifying analysis and space structure. Characteristic classes such as Chern, Pontryagin, and Stiefel–Whitney are obtained by integrating curvature forms on vector bundles, which, despite being locally defined, are indistinguishable as global topological entities. De Rham's theorem also explains topology holes through differential forms. Moreover, geometric evolution equations such as the Ricci flow, which contributed significantly to the solution of the Poincare conjecture by stabilizing the geometric structure of 3-manifolds, proved that topological classification is possible even by time-dependent metrics. Hence, it is concluded that there is a deep and dynamic interrelationship at the theoretical level between the in-situ analysis of dual geometry and the global properties of topology, which will continue to advance the understanding of the nature of manifolds mathematically.

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